The Fox–Li Operator as a Test and a Spur for Wiener–Hopf Theory

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Dedicated to the 80th Anniversary of Professor Stephen Smale

Abstract The paper is a concise survey of some rigorous results on the Fox–Li operator. This operator may be interpreted as a large truncation of a Wiener–Hopf operator with an oscillating symbol. Employing theorems from Wiener–Hopf theory one can therefore derive remarkable properties of the Fox–Li operator in a fairly comfortable way, but it turns out that Wiener–Hopf theory is unequal to the task of answering the crucial questions on the Fox–Li operator.

1 Masers, Lasers and the Fox–Li Operator

The story begun 50 years ago. Fox and Li [13] considered the repeated reflection of an electromagnetic wave of wave length λ between two plane-parallel rectangular mirrors. By a tensor product phenomenon, it suffices to suppose that the mirrors are infinite strips of height $2a$ with distance b between them. A distribution $u(x)$, $x \in (-a, a)$, of the field on one mirror goes over into the distribution given by

$$(Au)(x) = \frac{e^{i\pi/4}}{2\sqrt{\lambda}} \int_{-a}^{a} \kappa(x - y)u(y)dy, \quad x \in (-a, a), \tag{1}$$

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on the other mirror. Here κ is the function

$$\kappa(t) = \frac{e^{-ik\sqrt{t^2+b^2}}}{(t^2+b^2)^{1/4}}\left(1 + \frac{b}{\sqrt{t^2+b^2}}\right) \tag{2}$$

where $k = 1/\lambda$ denotes the wave number. What Fox and Li were interested in were the eigenvalues and eigenfunctions of the operator A: if $Au = \mu u$, then the distribution $u(x)$ will after n reflections be transformed into $\mu^n u(x)$. The number $1 - |\mu|^2$ is the energy loss of the mode u at one step. This setup is called a maser in the case of microwaves ($\lambda \approx 1$ cm) and a laser when working with light waves, in the range $\lambda \approx 5 \cdot 10^{-5}$ cm.

Let us consider the integral operator A given by (1) on $L^2(-a, a)$. Being compact, it has at most countably many eigenvalues with the origin as the only possible cluster point. Cochran [11] and Hochstadt [16] provided a rigorous argument which proves that A has at least one eigenvalue. However, there is no theorem that would imply more or anything else of interest about the operator A. Well, A has a difference kernel and hence one would expect that for large a the eigenvalues of A somehow mimic the values of the Fourier transform of κ,

$$\hat{\kappa}(\xi) := \int_{-\infty}^{\infty} \kappa(t) e^{i\xi t}\,dt, \quad \xi \in \mathbb{R}.$$

The function $\hat{\kappa}(\xi)$ is even, exponentially decaying as $|\xi| \to \infty$, and in $L^1(\mathbb{R})$. Had it been in $C(\mathbb{R})$, we would have had a theorem implying that the eigenvalues of A cluster along the range $\hat{\kappa}(\mathbb{R})$ as $a \to \infty$. However, $\hat{\kappa}(\xi)$ behaves like

$$\sqrt{\frac{\pi}{2b|\xi - k|}}\,[1 + i\,\mathrm{sign}(\xi - k)]$$

as $\xi \to k$ and hence it is not even in $L^\infty(\mathbb{R})$. In addition we should mention that the case $a \gg b$ is not the really interesting case in physics. One is therefore left with tackling the eigenvalue problem for A numerically, the big problem in this connection being that the kernel κ is highly oscillating: note that $k \approx 20{,}000\,\mathrm{cm}^{-1}$ for light waves.

Fox and Li found an ingenious way out. The physically relevant case is the one where $a \ll b$. They wrote

$$\exp(-ik\sqrt{t^2+b^2}) = \exp\left(-ikb\left(1 + \frac{t^2}{2b^2} + O\left(\frac{t^4}{b^4}\right)\right)\right),$$

and since $|t| < a$, one may ignore the O term if $kba^4/b^4 \ll 1$, that is, if $a^4 \ll \lambda b^3$. As $\lambda \ll b$, this assumption automatically implies that $a \ll b$, and therefore $(t^2 + b^2)^{1/4}$ and $b/\sqrt{t^2+b^2}$ may be replaced by $\sqrt{b}$ and 1, respectively. In summary, the operator A may be approximated by the operator

$$(A_1u)(x) = \frac{e^{i\pi/4}e^{-ikb}}{\sqrt{\lambda b}} \int_{-a}^{a} e^{-i(k/2b)(x-y)^2} u(y)dy, \quad x \in (-a, a).$$

The change of variables $x \to ax$, $y \to ay$ yields the operator

$$(A_2u)(x) = \frac{a e^{i\pi/4}e^{-ikb}}{\sqrt{\lambda b}} \int_{-1}^{1} e^{-i(ka^2/2b)(x-y)^2} u(y)dy, \quad x \in (-1, 1), \tag{3}$$

and abbreviating $\omega := ka^2/(2b) = a^2/(2\lambda b)$ and $\sqrt{i} := e^{i\pi/4}$ we arrive at the equality $A_2 = \sqrt{2\pi} e^{-ikb} \mathcal{F}_\omega^*$ with $\mathcal{F}_\omega^*$ and $\mathcal{F}_\omega$ defined on $L^2(-1, 1)$ by

$$(\mathcal{F}_\omega^* u)(x) = \sqrt{\frac{\omega i}{\pi}} \int_{-1}^{1} e^{-i\omega(x-y)^2} u(y)dy, \quad (\mathcal{F}_\omega u)(x) = \sqrt{\frac{\omega}{\pi i}} \int_{-1}^{1} e^{i\omega(x-y)^2} u(y)dy.$$

Note that $\mathcal{F}_\omega^*$ is really the adjoint of $\mathcal{F}_\omega$. The operator $\mathcal{F}_\omega$ is now called the Fox–Li operator, and the eigenvalues and eigenfunctions of this operator are what one wants to know.

After the change of variables $x \to x/\sqrt{\omega} - 1$, $y \to y/\sqrt{\omega} - 1$ the operator $\mathcal{F}_\omega$ becomes the operator given by

$$(F_\omega u)(x) = \frac{1}{\sqrt{\pi i}} \int_0^{2\sqrt{\omega}} e^{i(x-y)^2} u(y)dy, \quad x \in (0, 2\sqrt{\omega}), \tag{4}$$

on $L^2(0, 2\sqrt{\omega})$, and since $\omega = a^2/(2\lambda b)$ may also be assumed to be very large, F_ω is a very large truncation of a Wiener–Hopf operator.

In summary, the Fox–Li operator is a reasonable approximation to the original physical problem and at the same time a large truncated Wiener–Hopf operator whenever $\lambda^2 b^2 \ll a^4 \ll \lambda b^3$. Using the dimensionless parameters $\hat{a} := ka$ and $\hat{b} := kb$, these inequalities read $\hat{b}^{1/2} \ll \hat{a} \ll \hat{b}^{3/4}$, and ω becomes $\hat{a}^2/(2\hat{b})$. Fox and Li themselves showed that already the moderate choice $\hat{a} = 25$, $\hat{b} = 100$ leads to acceptable numerical results.

2 Wiener–Hopf Operators

An integral operator on $L^2(0, \infty)$ of the form

$$(Wu)(x) = \int_0^\infty \varrho(x - y)u(y)dy, \quad x \in (0, \infty),$$

is called a *Wiener–Hopf operator.* Such an operator is bounded on $L^2(0, \infty)$ if and only if the Fourier transform $a := \hat{\varrho}$, taken in the distributional sense, is a function in $L^\infty(\mathbb{R})$. The function ϱ is uniquely determined by its Fourier transform

a, henceforth we denote the operator W by $W(a)$. The function a is usually referred to as the *symbol* of $W(a)$. Note that $W(a)$ is the compression to $L^2(0,\infty)$ of the operator which acts on $L^2(\mathbb{R})$ by the following rule: take the Fourier transform, multiply the result by a, and then take the inverse Fourier transform.

For $\tau \in (0,\infty)$, the truncated Wiener–Hopf operator $W_\tau(a)$ is defined on $L^2(0,\tau)$ by

$$(W_\tau u)(x) = \int_0^\tau \varrho(x-y)u(y)\mathrm{d}y, \quad x \in (0,\tau). \tag{5}$$

The Fourier transform of $\varrho(t) = \mathrm{e}^{\mathrm{i}t^2}$ is $\hat{\varrho}(\xi) = \sqrt{\pi}\,\mathrm{i}\,\mathrm{e}^{-\mathrm{i}\xi^2/4}$. Thus, letting $\sigma(\xi) = \mathrm{e}^{-\mathrm{i}\xi^2/4}$, we see that the Fox–Li operator F_ω given by (4) is nothing but $W_{2\sqrt{\omega}}(\sigma)$, and the problem is to find the eigenvalues and eigenfunctions of $W_\tau(\sigma)$ as $\tau = 2\sqrt{\omega} \to \infty$.

The spectral theory of Wiener–Hopf operators is well developed, one could say that Wiener–Hopf operators and their discrete analogues, Toeplitz operators, are the best understood nontrivial classes of non-selfadjoint operators. We refer to [4] for a presentation of the matter. However, as already said, no result of this theory is immediately applicable to provide any deeper insight into the spectrum $\mathrm{sp}\, W_\tau(\sigma)$ of $W_\tau(\sigma)$. The best that is available to date is the following result.

Theorem 1. *We have* $\mathrm{sp}\, W(\sigma) = \overline{\mathbb{D}}$ *and* $\mathrm{sp}\, W_\tau(\sigma) \subset \overline{\mathbb{D}}$ *for every* $\tau > 0$, *where* $\overline{\mathbb{D}}$ *is the closed unit disc in the complex plane.*

This was established in [7]. The nontrivial part of the theorem is that $\mathrm{sp}\, W(\sigma)$ is all of $\overline{\mathbb{D}}$. In [7] it is actually shown that $\mathrm{sp}\, W_\tau(\sigma)$ is contained in the open unit disc $\mathbb{D}$ and that each point $\lambda \in \overline{\mathbb{D}}$ belongs to the essential spectrum of $W(\sigma)$, which means that $W(\sigma) - \lambda I$ is not even invertible modulo compact operators.

3 Eigenvalues

The physicists's intuition, like in Vainshtein's paper [23], and numerical computations, made by Cochran and Hinds [12] for probably the first time, indicate that the eigenvalues of $W_\tau(\sigma)$ lie along a spiral commencing at 1 and rotating clockwise to the origin: cf. Fig. 1. To date, no person alive has been able to prove this, even less so to derive rigourously the shape of the spiral. The following result gives an idea of what one is already proud of.

Theorem 2. *The operator* $W_\tau(\sigma)$ *is a trace class operator with at least one eigenvalue for every* $\tau > 0$, *and with the possible exception of at most countably many* $\tau \in (0,\infty)$, *the operator* $W_\tau(\sigma)$ *has a countable number of eigenvalues.*

This was proved in [11,16,19]. The approach of [11,16] is based on proving that $\det(I - zW_\tau(\sigma))$ is a nonconstant entire function of z. This function has infinitely many discrete zeros of finite multiplicity unless it reduces to a polynomial, which

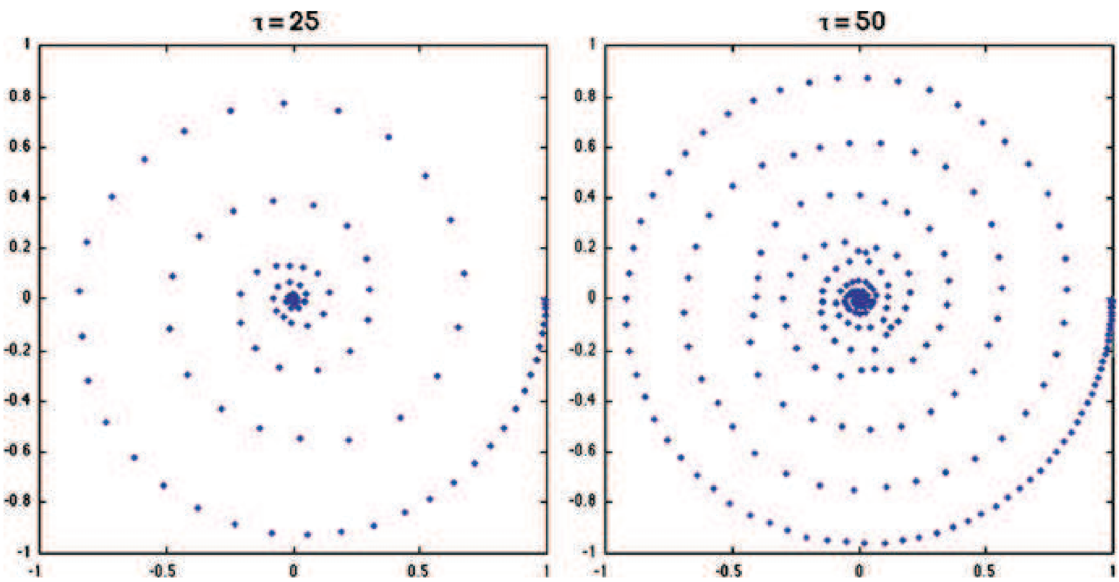

Fig. 1 The eigenvalues of $W_\tau(\sigma)$, with $\tau = 2\sqrt{\omega} = 25, 50$ and σ given by (4)

is shown to happen for at most countably many values of τ. Combining Theorem 1 with the observation that $W_\tau(a)$ is of trace class, one can say even a little more. Namely, let $\{\mu_n(W_\tau(\sigma))\}_{n=1}^N$ denote the eigenvalues of $W_\tau(\sigma)$ counted with their algebraic multiplicities. Then

$$\sum_{n=1}^{N} \mu_n(W_\tau(\sigma)) = \operatorname{tr} W_\tau(\sigma) = \frac{1}{\sqrt{\pi \mathrm{i}}} \int_0^\tau \mathrm{e}^{\mathrm{i}\cdot 0^2} \mathrm{d}x = \frac{\tau}{\sqrt{\pi \mathrm{i}}},$$

and since $|\mu_n(W_\tau(\sigma))| \le 1$ for all n, it follows that

$$\frac{\tau}{\sqrt{\pi}} = \left| \sum_{n=1}^{N} \mu_n(W_\tau(\sigma)) \right| \le \sum_{n=1}^{N} |\mu_n(W_\tau(\sigma))| \le N,$$

which reveals that $W_\tau(\sigma)$ has at least $\tau/\sqrt{\pi}$ eigenvalues.

Vainshtein [23] even raised a conjecture on the shape of the spiral.[1] It says that its parametric representation is $\mu = \exp(-\alpha(\tau)x^\nu - \mathrm{i}\beta(\tau)x^\nu)$, $x \in (0, \infty)$, with

$$\nu = 2, \quad \alpha(\tau) \approx \frac{\zeta(1/2)\pi^{3/2}}{8\sqrt{2}\,\tau^3}, \quad \beta(\tau) \approx \frac{\pi^2}{4\tau^2}, \tag{6}$$

where $\zeta(1/2)$ is Riemann's zeta function at the point $1/2$, and that $x = n$ gives approximately μ_n. We will return to this conjecture below.

[1]According to [12], this conjecture comes from "using a distinctly physical approach based on wave-guide theory", but we admit that we have not been able to follow the argument of [23]. Moreover, numerical computations do not support the conjecture.

Theorems of the type of Szegő's limit theorem [14] give asymptotic expansions for the trace $\operatorname{tr}\varphi(W_\tau(a)) = \sum_n \varphi(\mu_n(W_\tau(a)))$, where $\varphi : \mathbb{C} \to \mathbb{C}$ belongs to a certain class of so-called test functions. The following first-order result for the case $\varphi(z) = z^j$ was proved in [7].

Theorem 3. *For each fixed natural number j,*

$$\operatorname{tr} W_\tau^j(\sigma) = \frac{\tau}{\sqrt{\pi \mathrm{i} j}} + o(\tau) \quad \textit{as} \quad \tau \to \infty.$$

The operator $W_\tau^j(\sigma)$ is the integral operator on $L^2(0,\tau)$ with the kernel

$$m_j(x,y) := \frac{1}{(\pi \mathrm{i})^{j/2}} \int_0^\tau \dots \int_0^\tau \exp\left(\mathrm{i} \sum_{n=1}^{j} (x_n - x_{n+1})^2\right) \mathrm{d}x_2 \dots \mathrm{d}x_j,$$

where $x_1 = x$ and $x_{j+1} = y$. The trace of $W_\tau^j(\sigma)$ is $\int_0^\tau m_j(x,x)\mathrm{d}x$, and in [7] we proved that the leading term of the asymptotics of this multivariate oscillatory integral is $\tau/\sqrt{\pi \mathrm{i} j}$. We have not been able to determine the second term of the asymptotic expansion for general j.

Results like Theorem 3 can be used to test conjectures on the asymptotic eigenvalue distribution. Suppose we are given a family $\{b_\tau\}_{\tau>0}$ of functions $b_\tau{:}(0,\infty) \to \mathbb{C}$ and we want to know whether it might be true that the eigenvalues of $W_\tau(\sigma)$ are asymptotically distributed like samples of $b_\tau(x)$ at $x = n$. We have

$$\operatorname{tr} W_\tau^j(\sigma) = \sum_n \mu_n^j(W_\tau(\sigma)), \qquad \int_0^\infty b_\tau^j(x)\mathrm{d}x \approx \sum_n b_\tau^j(n),$$

and this is the motivation for saying that the eigenvalues of $W_\tau(\sigma)$ are asymptotically distributed as the values of b_τ (in a very weak sense) if, for each natural number $j \geq 1$,

$$\operatorname{tr} W_\tau^j(\sigma) = \int_0^\infty b_\tau^j(x)\mathrm{d}x + o(\tau) \quad \text{as} \quad \tau \to \infty.$$

Using Theorem 3 we showed the following theorem in [7], which justifies at least a few pieces of Vainshtein's conjecture.

Theorem 4. *Let $b_\tau(x) = \exp(-\alpha(\tau)x^\nu - \mathrm{i}\beta(\tau)x^\nu)$ with positive real numbers $\alpha(\tau)$, $\beta(\tau)$, ν. Then the eigenvalues of $W_\tau(\sigma)$ are asymptotically distributed as the values of b_τ if and only if*

$$\nu = 2, \quad \alpha(\tau) = o\left(\frac{1}{\tau^2}\right), \quad \beta(\tau) = \frac{\pi^2}{4\tau^2} + o\left(\frac{1}{\tau^2}\right).$$

4 Singular Values

The singular values of $W_\tau(\sigma)$ are the positive square roots of the eigenvalues of $W_\tau(\sigma)W_\tau^*(\sigma)$. Since $W_\tau^*(\sigma) = W_\tau(\overline{\sigma})$, we have

$$(W_\tau(\sigma)W_\tau^*(\sigma)u)(x) = \frac{1}{\pi}\int_0^\tau \left(\int_0^\tau e^{i(x-t)^2}e^{-i(t-y)^2}dt\right)u(y)dy, \quad x \in (0,\tau),$$

and hence $W_\tau(\sigma)W_\tau^*(\sigma) = V^*C_1V$ where V is the unitary operator given by $(Vu)(x) = e^{ix(\tau-x)}u(x)$ and C_1 is defined by

$$(C_1u)(x) = \frac{1}{\pi}\int_0^\tau \frac{\sin(\tau(x-y))}{x-y}u(y)dy, \quad x \in (0,\tau).$$

The change of variables $x \to x/\tau$, $y \to y/\tau$ shows that C_1 may be replaced by

$$(C_2u)(x) = \frac{1}{\pi}\int_0^{\tau^2} \frac{\sin(x-y)}{x-y}u(y)dy, \quad x \in (0,\tau^2).$$

The Fourier transform of $\sin t/(\pi t)$ is $\chi_{(-1,1)}$, the characteristic function of the interval $(-1,1)$. Consequently, the singular values of $W_\tau(\sigma)$ are the square roots of the eigenvalues of the operator $C_2 = W_{\tau^2}(\chi_{(-1,1)})$. This observation was probably first made in [6].

We are thus led to Wiener–Hopf with real-valued symbols. So, let us suppose that $a \in L^\infty(\mathbb{R})$ is real-valued. Then the operators $W(a)$ and $W_\tau(a)$ are selfadjoint. Hartman and Wintner [15] showed that $\operatorname{sp} W(a)$ equals the convex hull of the essential range of a. In [5] it was proved that $\operatorname{sp} W_\tau(a) \subset \operatorname{sp} W(a)$ for all $\tau > 0$ and that $\operatorname{sp} W_\tau(a)$ converges to $\operatorname{sp} W(a)$ in the Hausdorff metric. Using these general results and taking into account that $\|W_\tau(a)\| < \|a\|_\infty$ unless a is a constant, we arrive at the following.

Theorem 5. *The set of the singular values of $W_\tau(\sigma)$ is contained in $[0,1)$ for every $\tau > 0$ and converges to the segment $[0,1]$ in the Hausdorff metric as $\tau \to \infty$.*

Szegő's limit theorem gives the first term of the asymptotics of the trace of $\varphi(W_\tau(a))$ for arbitrary real-valued $a \in L^\infty(\mathbb{R})$ and the first two terms of the asymptotics if, in addition, a is smooth enough; see [4, 14]. Hence, for $a = \chi_{(-1,1)}$ we cannot derive a second order result in this way. Fortunately, the case where $a = \gamma\chi_{(\alpha,\beta)}$ was studied in detail by Landau and Widom [18].[2] They proved that if

[2] The reader might enjoy knowing the following, which is cited from [1]: "Harold Widom grew up in Brooklyn, New York. He went to Stuyvesant High School where he was captain of the math team. Coincidentally, the captain of the rival team at the Bronx High School of Science was Henry Landau ...".

$\alpha < \beta$ and $\gamma > 0$ are real numbers, then

$$\operatorname{tr}\varphi(W_\tau(\gamma\chi_{(\alpha,\beta)})) = \tau\,\frac{\varphi(\gamma)(\beta-\alpha)}{2\pi} + \frac{\log\tau}{\pi^2}\int_0^\gamma \frac{\gamma\varphi(x) - x\varphi(\gamma)}{x(\gamma - x)}\,\mathrm{d}x + O(1)$$

for every $\varphi \in C^\infty(\mathbb{R})$ satisfying $\varphi(0) = 0$. This was conjectured by Slepian [20]. A second proof of this result is in [24]. In [6] we applied this formula to $W_{\tau^2}(\chi_{(-1,1)})$ in order to get the following result on the finer distribution of the singular values of $W_\tau(\sigma)$.

Theorem 6. *Denote by $N(x, y)$ the number of singular values of $W_\tau(\sigma)$, counted with their multiplicities, which lie in the interval $(\sqrt{x}, \sqrt{y})$. Then for each δ in $(0, 1/2)$,*

$$N(1-\delta, 1) = \frac{\tau^2}{\pi} - \frac{2\log\tau}{\pi^2}\log\frac{1-\delta}{\delta} + o(\log\tau),$$

$$N(\delta, 1-\delta) = \frac{4\log\tau}{\pi^2}\log\frac{1-\delta}{\delta} + o(\log\tau),$$

$$N(0,\delta) = \infty.$$

Thus, although, by Theorem 5, the singular values fill [0, 1] densely as τ goes to ∞, the overwhelming majority of them are concentrated extremely close to the endpoints of the segment.

5 Complex Wave Numbers

Let us assume that the wave number k lies in the lower complex half-plane, $k = k_0 - \mathrm{i}\varepsilon$ with $k_0 = 1/\lambda$ and $\varepsilon > 0$. This assumption may not be of great interest in maser and laser theory, but it might be satisfied in problems of acoustics and, more importantly, it makes the problem nicely accessible to Wiener–Hopf theory.

Replacing k by $k_0 - \mathrm{i}\varepsilon$ in (2) and proceeding as in Sect. 1, the operator (3) now becomes

$$(A_{2,\varepsilon}u)(x) = \frac{a\mathrm{e}^{\mathrm{i}\pi/4}\mathrm{e}^{-\mathrm{i}kb}}{\sqrt{\lambda b}}\int_{-1}^{1} \mathrm{e}^{-\mathrm{i}(k_0a^2/2b)(x-y)^2}\mathrm{e}^{-(\varepsilon a^2/2b)(x-y)^2}u(y)\mathrm{d}y, \tag{7}$$

and letting $\omega = k_0a^2/(2b)$ and $\tau = 2\sqrt{\omega}$, we get the operator

$$(F_{\omega,\varepsilon}u)(x) = \frac{1}{\sqrt{\pi\mathrm{i}}}\int_0^\tau \mathrm{e}^{\mathrm{i}(x-y)^2}\mathrm{e}^{-(\varepsilon/k_0)(x-y)^2}u(y)\mathrm{d}y, \quad x \in (0,\tau) \tag{8}$$

in place of the operator (4). Here τ is a large number. The spectrum of (7) is what we are looking for, and this spectrum is $\sqrt{2\pi}\,\mathrm{e}^{-\mathrm{i}k_0 b}\mathrm{e}^{-\varepsilon b}$ times the complex conjugates of the points in the spectrum of $F_{\omega,\varepsilon}$. The Fourier transform of $(1/\sqrt{\pi\mathrm{i}})\mathrm{e}^{\mathrm{i}t^2}\mathrm{e}^{-(\varepsilon/k_0)t^2}$ is

$$\sigma_{\varepsilon/k_0}(\xi) = \frac{1}{\sqrt{1+\mathrm{i}\varepsilon/k_0}} \exp\left(-\frac{(\varepsilon/k_0)\xi^2}{4(1+\varepsilon^2/k_0^2)}\right) \exp\left(-\mathrm{i}\frac{\xi^2}{4(1+\varepsilon^2/k_0^2)}\right)$$

and hence we may write $F_{\omega,\varepsilon} = W_\tau(\sigma_{\varepsilon/k_0})$. Obviously, for $\varepsilon = 0$, the symbol σ_{ε/k_0} coincides with σ. The function σ is in $L^\infty(\mathbb{R})$ but not in $L^1(\mathbb{R})$, neither it is continuous on the one-point compactification $\dot{\mathbb{R}}$ of $\mathbb{R}$, which causes a great deal of problems in employing Wiener–Hopf theory. In contrast to this, σ_{ε/k_0} is in $L^1(\mathbb{R}) \cap C(\dot{\mathbb{R}})$, which facilitates matters significantly.

The kernels of the operators (4) and (8) are complex-symmetric, which implies that the symbol, i.e. the Fourier transform of the kernel function, is even. Note that if a is even, $a(\xi) = a(-\xi)$ for $\xi \in \mathbb{R}$, then we may think of the essential range $\mathcal{R}(a)$ of a as a curve which is traced out by $a(\xi)$ from $a(\infty)$ to $a(0)$ as ξ moves from $-\infty$ to 0 and then backwards from $a(0)$ to $a(\infty)$ as ξ moves further from 0 to $+\infty$. Complex-symmetric Toeplitz matrices and Wiener–Hopf operators with complex-symmetric kernels have certain peculiarities. The following was established in [6] and is the continuous analogue of results by Tilli [21] and Widom [25]. Namely, let $a \in L^1(\mathbb{R}) \cap C(\dot{\mathbb{R}})$, suppose a is even, and assume also that the essential range $\mathcal{R}(a)$ of a does not contain interior points. The last assumption is always satisfied if a has some minimal smoothness. Then the spectrum of $W_\tau(a)$ converges to $\mathcal{R}(a)$ in the Hausdorff metric. Secondly, if $\varphi : \mathbb{C} \to \mathbb{C}$ is any continuous function such that $\varphi(z)/z$ converges to a finite limit as $z \to 0$, then

$$\sum_n \varphi(\mu_n(W_\tau(a))) = \frac{\tau}{2\pi}\int_{-\infty}^{\infty} \varphi(a(\xi))\mathrm{d}\xi + o(\tau).$$

Applying these two general results to $a = \sigma_{\varepsilon/k_0}$, we obtain the following two theorems from [6].

Theorem 7. *As $\tau \to \infty$, the spectrum of $W_\tau(\sigma_{\varepsilon/k_0})$ converges in the Hausdorff metric to the logarithmic spiral*

$$\mathcal{R}(\sigma_{\varepsilon/k_0}) = \left\{ z \in \mathbb{C} : z = \frac{1}{\sqrt{1+\mathrm{i}\varepsilon/k_0}}\mathrm{e}^{-(\mathrm{i}+\varepsilon/k_0)\theta} \ \textit{for some} \ \ \theta \in [0,\infty] \right\}.$$

Theorem 8. *The number of eigenvalues of $W_\tau(\sigma_{\varepsilon/k_0})$ which lie close to the piece of the logarithmic spiral of the previous theorem given by $\theta \in (0, \theta_0)$ is*

$$\frac{2\tau}{\pi}\sqrt{(1+\varepsilon^2/k^2)\theta_0} + o(\tau).$$

Note that we are not able to prove something like these two theorems for $W_\tau(\sigma)$ because σ is neither in $L^1(\mathbb{R})$ nor in $C(\dot{\mathbb{R}})$.

6 Pseudospectrum

Fix $\varepsilon > 0$. The ε-pseudospectrum $\mathrm{sp}_\varepsilon B$ of a bounded linear operator B on some complex Hilbert space is the set of all $\mu \in \mathbb{C}$ for which $\|(B - \mu I)^{-1}\| \geq 1/\varepsilon$. The spectrum of B is considered to be a subset of $\mathrm{sp}_\varepsilon B$. If B is a normal operator, then $\mathrm{sp}_\varepsilon B$ is simply the closed ε-neigbourhood of $\mathrm{sp}\, B$. However, for non-normal operators this is in general no longer the case, and for such operators the pseudospectrum is in many instances of even greater use than the spectrum [22]. The notion of the psedospectrum was independently invented several times [22], and one of these inventions was made by Landau [17] when studying the Fox–Li operator. We first state a simple result from [7].

Theorem 9. *Given $\varepsilon > 0$, there is a $\tau_0 > 0$ such that $\mathrm{sp}_\varepsilon W_\tau(\sigma) \supset \overline{\mathbb{D}}$ for $\tau > \tau_0$.*

This theorem may be restated as follows. Given $\varepsilon > 0$ and $\mu \in \mathbb{D}$, there is a $\tau_0 > 0$ such that for every $\tau > \tau_0$ we can find $u_\tau \in L^2(0, \tau)$ satisfying $\|u_\tau\| = 1$ and $\|W_\tau(\sigma)u_\tau - \mu u_\tau\| \leq \varepsilon$. The following theorem is Landau's [17]. He takes μ from the unit circle $\mathbb{T}$ and is able to say much more in this case.

Theorem 10. *Given $\varepsilon > 0$, $\mu \in \mathbb{T}$, and $C > 0$, there exists a $\tau_0 > 0$ such that for every $\tau > \tau_0$ there are at least $C\tau$ functions $u_{\tau,n}$ which form an orthonormal system in $L^2(0, \tau)$ and satisfy $\|W_\tau(\sigma)u_{\tau,n} - \mu u_{\tau,n}\| \leq \varepsilon$. Moreover, if μ_1 and μ_2 are distinct points on $\mathbb{T}$, then these functions corresponding to μ_1 and μ_2 can be chosen to be mutually orthogonal.*

Landau [17] writes that this theorem "shows that for large Fresnel number ω the laser cannot be expected to settle to a single mode." Physical features of the pseudospectrum of the Fox–Li operator are also discussed in the work by Sir Michael Berry and his co-workers; see, e.g., [2, 3].

7 Challenges

So what are the big open problems for the Fox–Li operator we are, all progress notwithstanding, left with? Here are a few of them. (a) Determine the absolute value of the outmost or better of the outmost and next eigenvalues. (b) Prove that the eigenvalues cluster, in some sense, along a spiral. (c) Prove that this spiral migrates towards the unit circle as $\tau \to \infty$. (d) Determine the shape of the spiral. Is it as conjectured by Vainshtein (6), is it related to theta-three as tabled in [6], or is it something completely different? (e) Describe the density of the eigenvalue distribution along the spiral. (f) Determine the eigenfunctions: numerical indications in [10] are that the eigenfunctions corresponding to leading eigenvalues are trigonometric functions superimposed with low-amplitude rapid oscillation, while for small eigenvalues the eigenfunctions are wave packets.

These questions are of course also of interest for the operator with the original kernel function (2).

We should emphasize that, with the exception of problems (d) and (e), these questions have all been solved numerically. Approaching the Fox–Li operator numerically is not a triviality, since this involves working with highly oscillatory integrals. That Cochran and Hinds [12] were able to show us the spirals as early as 1974 must in this light be appreciated as an admirable feat. Since then numerical methods for highly oscillatory integral equations have been elaborated by many mathematicians, and by now the apparatus is well developed to overcome nearly all subtleties caused by high frequencies. We refer to the recent papers [9, 10] and the references therein for more on the computational mathematics for the Fox–Li and related operators.

Finally, we repeat that two peculiarities of the Fox–Li operator are that its kernel is complex-symmetric and that it depends only on the difference of the arguments. To gain deeper insight into the Fox–Li operator it seems therefore reasonable first to attain greater command of simpler operators with such kernels. In [8], we accordingly considered Wiener–Hopf operators with even and rational symbols. These are given by (5) where $\varrho(t)$ is a finite sum of terms of the form $p_n(|t|)e^{-\gamma_n|t|}$ with polynomials p_n and complex numbers γ_n such that $\operatorname{Re}\gamma_n > 0$. The symbol $a = \hat{\varrho}$ is an even and rational function in $L^1(\mathbb{R}) \cap C(\dot{\mathbb{R}})$. Hence, by what was outlined in Sect. 5, $\operatorname{sp} W_\tau(a)$ converges to the curve $\mathcal{R}(a)$ formed by the range of a in the Hausdorff metric. However, in the case at hand we can say more. There are explicit formulae for the Fredholm determinants of Wiener–Hopf operators with rational symbols. Given a and under additional technical assumptions, we used these formulae to construct a certain function $b:(0,\infty) \to \mathbb{C}$ and to prove that there is a numbering $\{\mu_n\}_{n=1}^{\infty}$ of the eigenvalues of $W_\tau(a)$ such that, with $\xi_n := n\pi/\tau$,

$$\mu_n = a(\xi_n) + \frac{1}{2\tau}a'(\xi_n)\arg b(\xi_n) - \frac{i}{2\tau}a'(\xi_n)\log|b(\xi_n)| + O(1/\tau^2).$$

Note that the tangent to $\mathcal{R}(a)$ through $a(\xi_n)$ has the parametric representation $\mu = a(\xi_n) + a'(\xi_n)t$, $t \in \mathbb{R}$, and increasing values of the parameter t provide the tangent with an orientation. The point $a(\xi_n) + (1/2\tau)a'(\xi_n)\arg b(\xi_n)$ lies on this tangent. It follows that, up to the $O(1/\tau^2)$ term, the eigenvalue μ_n is located on the right of the tangent if $|b(\xi_n)| > 1$, while it is on the left of the tangent if $|b(\xi_n)| < 1$. Furthermore, the eigenfunctions for an eigenvalue μ_n are shown to be linear combinations of $e^{iz_j x}$ where $z_j \in \mathbb{C}$ ranges over the finite set of solutions of the algebraic equation $a(z) = \mu_n$.

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