Asymptotics of all eigenvalues of large non self adjoint Toeplitz matrices

Sergei Grudsky

CINVESTAV del I.P.N, México D.F, México

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Abstract

The asymptotic behavior of the spectrum of large Toeplitz matrices has been studied for almost one century now. Among this huge work, we can find the Szegö theorems on the eigenvalue distribution and the asymptotics for the determinants, as well as other theorems about the individual asymptotics for the smallest and largest eigenvalues.

The first results about uniform individual asymptotics for all the eigenvalues and eigenvectors appeared in 2008–2009. The most of the results in this area are concerned to self–joint case. These talk is devoted to some cases of the non self–joint Toeplitz matrices.

Main object.

Spectral properties of larger finite Toeplitz matrices

$$A_n = (a_{j-k})_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \dots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \dots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}.$$

$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j,\ t \in \mathbb{T}$-symbol of $\{A_n\}_{n=1}^{\infty}$

Eigenvalues, eigenvectors singular values, condition numbers, invertibility and norms of inverses, e.t.c.

$n \sim 1000$ is a business of numerical linear algebra.

Statistical physics - $n = 10^7 - 10^{12}$ - is a business of asymptotic theory.

Levels of asymptotics properties the large Toeplitz matrices

I. Szegö's (first limit) Theorem

Theorem (1)

Let $a \in L_\infty$ and let $\Omega \in \mathbb{C}$ an open set contains convex hull of image $a(t)(t \in \mathbb{T})$. If f is analytic in Ω, then

$$\frac{1}{n} tr\, f\left(T_n(a)\right) := \frac{1}{n}\sum_{j=1}^{n} f\left(\lambda_n^{(j)}\right) \to \frac{1}{2\pi}\int_0^{2\pi} f\left(a\left(e^{i\varphi}\right)\right) d\varphi.$$

II. Limit set of eigenvalues.

III. Individual asymptotics of eigenvalues.

Limit of eigenvalues

$$S_p\left(T_n(a)\right) = \bigcup^{n}\left\{\lambda_n^{(j)}\right\}$$

$$\Lambda(a) = \limsup_{n\to\infty} S_p\left(T_n(a)\right)$$

I. Self–joint case: $a(t) = \bar{a}(t)$, $a(t) \in L_\infty$,

$$\Lambda(a) = [\operatorname{ess}\inf_{t\in\mathbb{T}} a(t), \operatorname{ess}\sup_{t\in\mathbb{T}} a(t)].$$

II. $\mathcal{R}(a) = \left\{z \in \mathbb{C} \middle| z = a(t),\ t \in \mathbb{T}\right\}$. Let $\mathcal{R}(a)$ is curve without interior, then

$$\Lambda(a) = \mathcal{R}(a)$$

III. $\mathcal{R}(a)$–smooth curve (with interior) with several points of power singularities, then

$$\Lambda(a) = \mathcal{R}(a)$$

IV. Polynomial case.

Theorem (2)(Schmidt and Spitzer, 1961)

Let $a(t) = \sum_{k=-\tau}^{s} a_k t^k \, (r \geq 1,\ s \geq 1,\ a_s \neq 0,\ a_{-\tau} \neq 0)$ *be a Laurent polynomial, let* $z_1(\lambda), z_2(\lambda), \ldots, z_{r+s}(\lambda)$ *be the zeros of the polynomial* $z^r(a(t) - \lambda)$, *labeled so that*

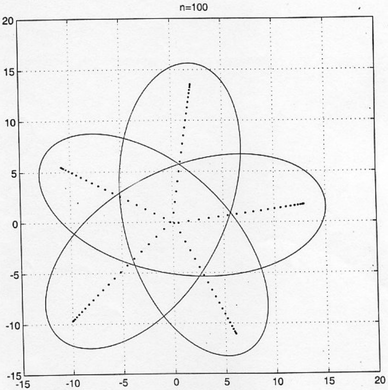
$$|z_1(\lambda)| \leq |z_2(\lambda)| \leq \ldots \leq |z_{r+s}(\lambda)|.$$

Then

$$\Lambda(a) = \{\lambda \in \mathbb{C} | |z_r(\lambda)| = |z_{r+1}(\lambda)|\} .$$

$$a(t) = t^{-4} - \frac{i}{10} t^{-3} + (i+5) t^{-2} - \frac{i}{2} t^{-1} + \frac{i}{2} t^{2} + (10+i) t^{3} - \cancel{i t^{4}}$$

?



I. Two parameters:

n- dimensions of matrices;

j- number of eigenvalue

$$1 \leq j \leq n$$

Asymptotics by n uniformly in j.

II. Distance between λ_j and λ_{j+1} is small:

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n}\right) - \text{normal case}$$

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n^\gamma}\right) - \text{special case}$$

$$\lambda_j = \lambda_{j+1} \qquad \text{– exceptional case}$$

Publications about asymptotics of individual eigenvalues. Self–joint case.

1. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Inside the Eigenvalues of Certian of Hermitian Toeplitz Band Matrices. Computational and Applied Mathematics, 233 (2010), 2245-2264 pp.

2. Deift P, Its A, and Krasovsky I. Eigenvalues of Toeplitz matrices in the bulk of the spectrum. Bulletin of the Institute of Mathematics Academia Sinica (New Series) 7 (2012), 437-461 pp.

3. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvalues of Hermitian Toeplitz matrices with smooth simple-loop symbols. Journal of Mathematical Analysis and Applications Volume 422, Issue 2, 15 February 2015, 1308-1334 pp.

4. J.M. Bogoya, S.M. Grudsky and E.A. Maksimenko.. Eigenvalues of Hermitian Toeplitz Matrices Generated by Simple-loop Symbols with Relaxed Smoothness. Operator Theory: Advances and Applications, Volume 259, 2017, 179–212 pp.

Main results-self–joint case

For $\alpha \geq 0$, we denote by W^α the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$\|a\|_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^\alpha < \infty.$$

Put $g(\varphi) := a(e^{i\varphi}),\ \varphi \in [0, 2\pi]$.

(I) a is real-valued;

(II) the range of g is a closed interval $[0, \mu]$ with $\mu > 0$, $g(0) = g(2\pi) = 0$, $g''(0) = g''(2\pi) > 0$, there is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = \mu$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

Symbols in the class SL^{α} are known as *simple-loop symbols*. (I) is equivalent to the condition that all matrices $T_n(a)$ ($n \in \mathbb{Z}_+$) are Hermitian (self-adjoint). If $a \in W^{\alpha}$, then $g \in C^{\lfloor \alpha \rfloor}[0, 2\pi]$ where $\lfloor \alpha \rfloor$ is the integer part of α. So, the condition $a \in \mathrm{SL}^{\alpha}$ with $\alpha \geq 1$ implies, in particular, that g belongs to $C^1[0, 2\pi]$.

In this work, for every $\alpha \geq 1$, we introduce a new class of symbols MSL^α (the modified simple loop class). Namely, $a \in \mathrm{MSL}^\alpha$ if $a \in \mathrm{SL}^\alpha$ and

(III) there exist functions $q_1, q_2 \in W^\alpha$ satisfying

$$a(t) = (t-1)q_1(t) \quad \text{and} \quad a(t) - a(e^{i\varphi_0}) = (t - e^{i\varphi_0})q_2(t). \qquad (1)$$

It is easy to proof that, if $a \in W^\alpha$, then q_1 and q_2 both belong to $W^{\alpha-1}$, but we require the stronger condition (III) instead.

Symmetric case

$$g(s) = g(2\pi - s).$$

Let further $\lambda = g(s)$

$$\beta(\sigma, s) := \frac{(g(\sigma) - g(s))e^{is}}{(e^{i\sigma} - e^{is})(e^{-i\sigma} - e^{-is})} = \frac{g(s) - g(\sigma)}{4 \sin \frac{\sigma - s}{2} \sin \frac{\sigma - s}{2}}.$$

We will show that β is a continuous and positive function on $[0, 2\pi] \times [0, \pi]$. We define the function $\eta : [0, \pi] \to \mathbb{R}$ by

$$\eta(s) := \theta(\psi(s)) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - s}{2}} \, d\sigma - \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - s}{2}} \, d\sigma,$$

the integrals taken in the principal-value sense.

Theorem (3)

Let $\alpha \geq 1$ *and* $a \in \mathrm{MSL}^{\alpha}$. *Then for every* $n \geq 1$*:*

(I) *the eigenvalues of* $T_n(a)$ *are all distinct:* $\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}$;

(II) *the numbers* $s_j^{(n)}$, *such that* $\lambda_j^{(n)} = g(s_j^{n})$ $(j = 1, \ldots, n)$ *satisfy*

$$(n+1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j), \tag{2}$$

where $\Delta_1^{(n)}(j) = o\left(\frac{1}{n^{\alpha-1}}\right)$ *as* $n \to \infty$, *uniformly in* j;

(III) *for every sufficiently large* n, (2) *has exactly one solution* $s_j^{(n)} \in [0, \pi]$ *for each* $j = 1, \ldots, n$.

Theorem (4)

Let $d_j^{(n)} = \dfrac{\pi j}{n+1}$*, then under the conditions of previous Theorem*

$$\lambda_j^{(n)} = g(d_j^{(n)}) + \sum_{k=1}^{\lfloor\alpha\rfloor} \frac{r_k(d_j^{(n)})}{(n+1)^k} + \Delta_3^{(n)}(j),$$

where $\Delta_3^{(n)}$ *is* $o\Big(\frac{1}{n^\alpha}(d_j^{(n)}(\pi - d_j^{(n)}))^{\alpha-1}\Big)$ *if* $1 \le \alpha < 2$ *and* $o\Big(\frac{d_j^{(n)}}{n^\alpha}(\pi - d_j^{(n)})\Big)$ *if* $\alpha \ge 2$ *as* $n \to \infty$*, uniformly in* j*. The coefficients* r_k *can be calculated explicitly; in particular,*

$$r_1(s) = -\psi'(s)\eta(s) \quad \text{and} \quad r_2(s) = \frac{1}{2}\psi''(s)\eta^2(s) + \psi'(s)\eta(s)\eta'(s).$$

Inner eigenvalue

Symmetric symbol: $\psi(\varphi) = g(\varphi) = (a(e^{i\varphi}))$

Normal case: $g'(\varphi) \neq 0$ ($\varepsilon < \frac{\pi j}{n+1} < \pi - \varepsilon$). Inner eigenvalues

$$\lambda_j^{(n)} = g\left(\frac{\pi j}{n+1}\right) + \frac{c_1\left(\frac{\pi j}{n+1}\right)}{n+1} + O\left(n^{-2}\right).$$

Distance between next eigenvalues is

$$O\left(\frac{1}{n}\right)$$

Extreme eigenvalue

Exceptional case: $\left(\frac{\pi j}{n+1}\right) \leq \varepsilon$.

$$\lambda_j^{(n)} = g\left(\frac{\pi j}{n+1}\right) + O\left(\frac{1}{n^3}\right), \quad \frac{\pi j}{n+1} \leq \varepsilon.$$

Distance between next eigenvalues is

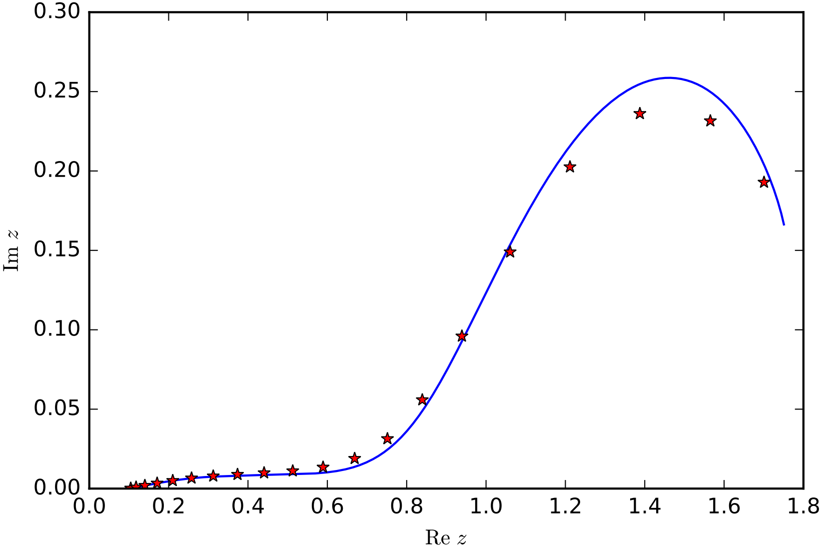
$$O\left(\frac{1}{n^2}\right)$$

Symmetric complexvalue symbol

$$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \; a_j = a_{-j}.$$

$$a_1(t) = c_1 \sin(c_0 t^2) + \frac{1}{20}\left((1+t)^{5/2} + (1-t)^{5/2}\right), \; t \in [-\pi; \pi],$$

$$c_0 = \frac{1}{5} - \frac{1}{6}i, \; c_1 = \frac{(1-\pi)^{3/2} - (\pi+1)^{3/2}}{16\pi c_0 \cos(\pi^2 c_0)}$$


0.30
0.25
0.20
0.15
0.10
0.05
0.00
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
Im z
Re z

A.A. Batalshchikov, S.M. Grudsky, V.A. Stukopin. Asymptotics of eigenvalues of symmetric Toeplitz band matrices. Linear Algebra and its Applications Volume 469, 15, 15 March 2015, 464-486 pp.

$$a(t) \in W^{\alpha},\ \alpha \geq 2,\ g(\varphi) = a\left(e^{i\varphi)}\right),\ \varphi \in [0, 2\pi].$$

$$\|a\|_{\alpha} := \sum_{j=-\infty}^{\infty} |a_j| \left(|j| + 1\right)^{\alpha} < \infty.$$

I. $g(\varphi) = g(2\pi - \varphi)$ – symmetric.

II. $\mathcal{R}(a)$ – is simple curve without self–intersection
$\mathcal{R}(a) = (M_0, M_1),\ M_0 \neq M_1.$

III. $g'(\varphi) \neq 0,\ \varphi \in (0, \pi).$

IV. $g''(0) = g''(2\pi) \neq 0,\ g''(\pi) \neq 0.$

Main Idea

$$T_n(a) = T_n(a_n) \quad \textbf{!}$$

$$a_n(t) = \sum_{j=-(n-1)}^{n-1} a_j t^j,\ g_n(\psi) = a_n\left(e^{i\psi}\right),$$

where

$$\psi \in \Pi_n := \left\{\psi = \varphi + i\delta | \varphi \in [c/n,\ \pi - c/n],\ \delta \in [-C/n,\ C/n]\right\}$$

with $c > 0$ and $C > 0$.

$$\mathcal{R}_n(a) =: \left\{g_n(\psi) : \psi \in \Pi_n\right\}$$

Lemma (1)

Let $a(t) \in W^\alpha$, $\alpha \geq 2$, and satisfies condition 1.-2.-3.-4. Then the map

$$g_n(\psi) : \Pi_n \to \mathcal{R}_n(a) \text{ is bijection for } n \text{ large enough.}$$

Lemma (2)

Let $\psi \in \Pi_n$, $a(t) \in W^\alpha$, $\alpha \geq 0$, and $m = [\alpha]$. Then

$$g_n(\psi) = g(\varphi) + \sum_{k=1}^{m} \frac{g^{(k)}(\varphi)}{k!}(i\delta)^k + \sum_{k=0}^{m+1} \alpha_{n,k}(\varphi)(i\delta)^k,$$

where $\alpha_{n,k}(\varphi) \in W^0$, and

$$\|\alpha_{n,k}\|_0 = o(n^{k-\alpha}),\ k = 0, 1, \ldots, m$$

and

$$\|\alpha_{n,m+1}\|_0 = O(n^{m+1-\alpha}).$$

Let $g_n(\varphi_{1,n}(\lambda)) = \lambda$, then introduce following functions

$$\hat{b}_n(t,\lambda) = \frac{(a_n(t) - \lambda)e^{i\varphi_{1,n}(\lambda)}}{\left(t - e^{i\varphi_{1,n}(\lambda)}\right)\left(t^{-1} - e^{i\varphi_{1,n}(\lambda)}\right)}, \ \lambda \in \mathcal{R}_n(a),$$

$$\theta_n(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log \hat{b}_n(\tau,\lambda)}{\tau - e^{i\varphi_{1,n}(\lambda)}} d\tau - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log \hat{b}_n(\tau,\lambda)}{\tau - e^{-i\varphi_{1,n}(\lambda)}} d\tau, \ \lambda \in \mathcal{R}_n(a)$$

$$\eta_n(s) := \theta_n(g_n(s)), \ s \in \Pi_n.$$

Consider two sequences

$$d_{j,n} = \frac{\pi j}{n+1}, \ e_{j,n} = d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1}.$$

Introduce small domains

$$\Pi_{j,n}(a) := \left\{ s \in \Pi_n(a), \ |s - e_{j,n}| \le \frac{c_n}{n+1} \right\},$$

where $c_n \to 0$.

Theorem (5)

Let $a \in \mathrm{CSL}^{\alpha}$, $\alpha \geq 2$. *Then for sufficiently large natural number* n *the following statements hold:*

1. *all eigenvalues* $T_n(a)$ *are different, and* $\lambda_{j,n} \in g(\Pi_{j,n}(a))$ *for* $j = 1,\ 2,\ \dots,\ n$
2. *values* $s_{j,n}$ *such that* $\lambda_{j,n} = g_n(s_{j,n})$ *satisfy the equation*

$$(n+1)s + \eta_n(s) = \pi j + \Delta_n(s), \qquad j = 1,\ 2,\ \dots,\ n \tag{3}$$

   with $|\Delta_n(s)| = o(1/n^{\alpha-2})$ *where* $\Delta_n(s) \to 0$ *as* $n \to \infty$ *uniformly respect to* $s \in \Pi_n(a)$.
3. *Equation (3) have a unique solution in the domain* $\Pi_{j,n}(a)$.

Theorem (6)

Under conditions of Theorem 5

$$\lambda_j^{(n)} = g(d_{j,n}) + \sum_{k=1}^{[\alpha]-1} \frac{r_k(d_{j,n})}{(n+1)^k} + \Delta_3^{(n)}(j) \tag{4}$$

where

$$\Delta_3^{(n)}(j) = \begin{cases} o\left(\frac{d(\pi-d)}{n}\right), & \alpha = 2, \\ O\left(\frac{d(\pi-d)}{n^{\alpha-1}}\right), & \alpha > 2. \end{cases}$$

as $n \to \infty$ *uniformly in* j *and* $d = d_{j,n}$. *The coefficients* r_k *can be calculated explicitly; in particular*

$$r_1(\varphi) = -g'(\varphi)\eta(\varphi) \quad \text{and} \quad r_2(\varphi) = \frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi).$$

Remark

For small j, $\left(j/n^2 \to 0\right)$ *we have following asymptotics from Theorem 6, for* $\alpha \geq 3$,

$$\lambda_{j,n} = g(0) + C_3 \frac{j^2}{(n+1)^2} + O\left(\frac{j^2}{(n+1)^2}\right),$$

where

$$C_3 = \frac{\pi^2 g''(0)}{2}.$$

Location the eigenvalue relative to $\mathcal{R}(a)$

$$\lambda_{j,n} = g(d_{j,n}) - \frac{g'(d_{j,n})\,\eta(d_{j,n})}{n+1} + O\left(\frac{1}{n^2}\right).$$

Let $\widetilde{e_{j,n}} = d_{j,n} + \dfrac{\operatorname{Re}\eta(d_{j,n})}{n+1}$, then

$$\lambda_{j,n} = g(\widetilde{e_{j,n}}) + i\frac{g'(\widetilde{e_{j,n}})\,(\operatorname{Im}\eta(d_{j,n}))}{n+1} + O\left(\frac{1}{n^2}\right).$$

$\lambda_{j,n}$ is located on the normal to curve $\mathcal{R}(a)$ in the point $z = g(\widetilde{e_{j,n}})$ with exactitude to $O\left(\frac{1}{n^2}\right)$

$$\frac{|g'(e_{j,n})\operatorname{Im}\eta(d_{j,n})|}{n+1} + O\left(\frac{1}{n^2}\right) - \text{distance between } \lambda_{j,n} \text{ and } \mathcal{R}(a).$$

Numerical example

Approximations:

$$\lambda_{1,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1},$$
$$\lambda_{2,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1} + \frac{\frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi)}{(n+1)^2}.$$

Errors:

$$\Delta_1^{(n)} = \max_j \left| \frac{\lambda_{1,j} - \lambda_j}{\lambda_j} \right|,$$
$$\Delta_2^{(n)} = \max_j \left| \frac{\lambda_{2,j} - \lambda_j}{\lambda_j} \right|,$$

$$a_1(t) \in W^{2,5-\delta}, \ \forall \quad \delta > 0$$

n	20	40	80	160	320
$\Delta_1^{(n)}$	3.2e-03	8.8e-04	2.3e-04	5.9e-05	1.5e-05
$\Delta_2^{(n)}$	3.9e-04	5.6e-05	7.2e-06	9.2e-07	1.2e-07

Symbols with Fisher–Harturg singularity.

$$a_{\alpha,\beta}(t) = (1-t)^{\alpha}(-t)^{\gamma}, \qquad 0 < \alpha < |\beta| < 1.$$

Conjecture of
H.Dai, Z.Geary and L.P.Kadanoff, 2009

$$\lambda_j^{(n)} \sim a_{\alpha,\beta}\left(\omega_j \cdot \exp\left\{(2\alpha+1)\,\frac{\log}{n}\right\}\right),$$

where $\omega_j = \exp\left(-i\,\frac{2\pi j}{n}\right)$.

1. H. Dai, Z. Geary, L.P. Kadanoff. Asymptotics of eigenvalues and eigenvectors of Toeplitz matrices. Journal of Statistical Mechanics: Theory and Experiment, May, 2009, PO5012.

2. J. M. Bogoya, Albrecht Böttcher and Sergei M. Grudsky. Asymptotics of individual eigenvalues of a class of large Hessenberg Toeplitz matrices. Operator Theory: Advances and Applications, 220 (2012), 77-95 pp.

Complex value case

$$a(t) = t^{-1}(1-t)^{\alpha} f(t), \quad \alpha \in R_+ \backslash N$$

where

1. $f(t) \in H^{\infty} \cap C^{\infty}$.
2. f can be analytically extended to a neighborhood of $\mathbb{T} \backslash \{1\}$.
3. The range of the symbol a $\mathcal{R}(a)$ is a closed Jordan curve without loops and winding number -1 around each interior point.

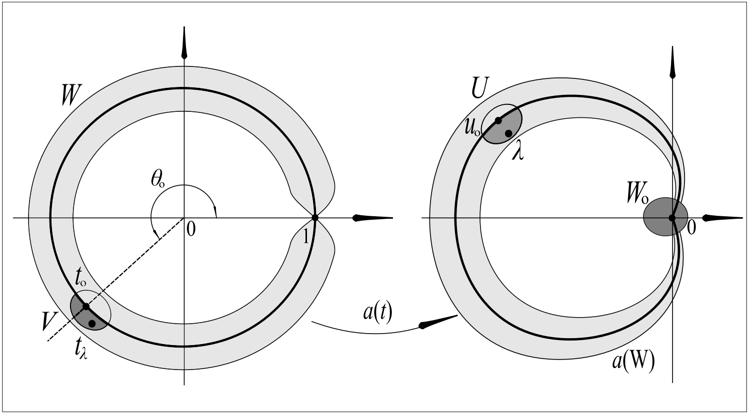


Figure: The map $a(t)$ over the unit circle.

Lemma (3)

Let $a(t) = t^{-1}h(t)$ be a symbol that satisfies the following conditions:

1. $h \in H^\infty$.
2. $\mathcal{R}(a)$ *is a closed Jordan curve in* $\mathbb{C}$ *without loops.*
3. $wind_\lambda(a) = -1$, *for each* λ *in the interior of sp* $T(a)$.

Then, for each λ *in the interior of sp* $T(a)$, *we have the equality*

$$D_n(a - \lambda) = (-1)^n h_o^{n+1} \left[\frac{1}{h(t) - \lambda t} \right]_n ,$$

for every $n \in \mathbb{N}$.

Theorem (7)

We have the following asymptotic expression for λ_j:

$$\lambda_j = a(\omega_j) + (\alpha + 1)\omega_j a'(\omega_j)\frac{\log(n)}{n} + \frac{\omega_j a'(\omega_j)}{n}\log\left(\frac{a^2(\omega_j)}{c_o a'(\omega_j)\omega_j^2}\right) + \mathcal{O}\left(\frac{\log(n)}{n}\right)^2, \ n \to \infty,$$

where $\omega_j = \exp\left(-i\,\frac{2\pi j}{n}\right)$.

$n = 4096$

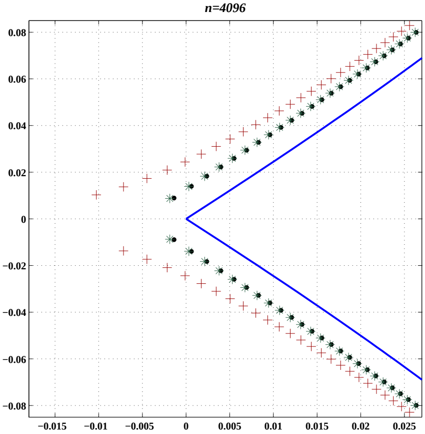


Figure: The solid blue line is the range of a. The black dots are sp $T_n(a)$ calculated by *Matlab*. The red crosses and the green stars are the approximations, for 1 and 2 terms respectively. Here we took $\alpha = 3/4$.