On the Trace-Class Property of Hankel Operators Arising in the Theory of the Korteweg-de Vries Equation

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Abstract

The trace-class property of Hankel Operators (and their derivatives with respect to the parameter) with strongly oscillating symbol is studied. The approach used is based on Peller's criterion for the trace-class property of Hankel operators and on the precise analysis of the arising tripe integral using the saddle-point method. Apparently, the obtained results are optimal. They are used to study the Cauchy problem for the Korteweg-de Vries equation. Namely, a connection between the smoothness of the solution and the rate of decrease of the initial data at positive infinity is established.

Hankel Operators

$$\mathbb{H}(\varphi_x) := JP^-\varphi_x P^+ : H^2(\Pi) \to H^2(\Pi), \tag{1}$$

where $H^2(\Pi)$ is Hardy space in the upper half-plane

$$\Pi := \{\lambda \in \mathbb{C} \,|\, \mathrm{Im}\,\lambda > 0\}\,;$$

J - is the reflection operator defined by:

$$(Jf)(\lambda) = f(-\lambda),\ \lambda \in \mathbb{R},$$

and $P^{\pm}$ are the analytic projections defined by

$$\left(P^+f\right)(\xi) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f(\tau)}{\tau - \xi}\, d\tau, \quad \xi \in \overline{\Pi},$$

$$\left(P^-\varphi\right)(\xi) = \left(JP^+J\varphi\right)(\xi),$$

which act on the space $L_2(\mathbb{R})$.

Note that is ξ belongs to the real axis $\mathbb{R}$, then the above integral is understood as the limit value almost everywhere over non-tangential directions in the upper half-plane Π.

Symbol of Hankel Operator

$$\varphi_x(\lambda) = T(\lambda)\ G_-(\lambda)\ e^{i\Phi(\lambda,x)}. \tag{2}$$

Here

$$\Phi(\lambda, x) = 8t\lambda^3 + 2x\lambda,\ \ t > 0,\ \ x \in \mathbb{R}. \tag{3}$$

The function $G_-(\lambda)$ can be represented as the Fourier integral over the half-axis:

$$G_-(\lambda) = \int_0^\infty e^{-2i\lambda s} g(s)\, ds, \tag{4}$$

where $g(s) \in L_1\left(\mathbb{R}_+, (1+s)^\alpha\right)$, is nonegative-valued almost everywhere, i.e.

$$\int_0^\infty g(s)(1+s)^\alpha ds < \infty, \quad \alpha \geq 0. \tag{5}$$

$$T(\lambda) \in H^\infty(\Pi).$$

Main Result

Let $\mathfrak{S}_1$ denote the set of all trace-class operators acting on the space $H^2(\Pi)$. Recall that a compact operator A belong to $\mathfrak{S}_1$, if the sequence of its singular numbers $\{s_j(A)\}_{j=1}^{\infty}$ is summable. The norm of an operator A in $\mathfrak{S}_1$ is defined as

$$\|A\|_{\mathfrak{S}_1} := \sum_{j=1}^{\infty} |s_j(A)|.$$

Along with the operator (1) we consider its derivatives with respect to the parameter x. It is easy to see that

$$\frac{\partial^j}{\partial x^j} \mathbb{H}(\varphi_x) = \mathbb{H}(\varphi_{j,x}), \tag{6}$$

where

$$\varphi_{j,x}(\lambda) = (2i)^j \lambda^j \varphi_x(\lambda), \quad j = 0, 1, 2, \ldots . \tag{7}$$

If $\varphi \in L^{\infty}(R) => \mathbb{H}(\varphi)$ is bounded on $H^2(\Pi)$

$$\mathbb{H}(h - \varphi_0) = H(\varphi_0),\ h \in H^{\infty}(\Pi).$$

It should be noted that φ and h could be unbounded.

Theorem

If the function $\varphi_x(\lambda)$ is of the form (2)-(5) with $g(s) \in L_1\left(\mathbb{R}_+(1+s)^{j/2}\right)$, $j \in \mathbb{N}$, *then*

$$\frac{\partial^k}{\partial x^k}\mathbb{H}(\varphi_x) \in \mathfrak{S}_1, \quad k = 0, 1, \ldots j,$$

and

$$\left\|\frac{\partial^k}{\partial x^k}\mathbb{H}(\varphi_x)\right\|_{\mathfrak{S}_1} \leq \begin{cases} L_1, & x > 0 \\ L_2\left(1+|x|\right)^{k/2}, & x < 0, \end{cases}$$

where the constants L_1 and L_2 are independent of $x \in \mathbb{R}$.

Peller's Theorem

We say that a function $f(\xi)$ analytic in Π belongs to the space $A_1^1(\Pi)$ if and only if

$$\|f\|_{A_1^1(\Pi)} := \int_0^\infty \int_{-\infty}^\infty |f''(\xi_1 + i\xi_2)|\, d\xi_1\, d\xi_2 + \sup\{f(\xi)|\xi_2 \geq 1\} < \infty,$$

where $\xi = \xi_1 + i\xi_2$ is a complex variable belonging to the complex plane $\mathbb{C}$. we introduce the following modification of an analytic projection:

$$\left(\widetilde{P^+} f\right)(\xi) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\tau - \xi} - \frac{\tau}{1 + \tau^2} \right) f(\tau)\, d\tau.$$

Theorem (V.Peller, 1980)

Let $\varphi \in L_\infty(\mathbb{R})$, *Then* $\mathbb{H}(\varphi) \in \mathfrak{S}_1$ *if and only if*

$$\left(\widetilde{P^+\overline{\varphi}}\right)(\xi) \in A_1^1(\Pi).$$

Lemma

Let $\varphi = h\varphi_1$, *where* $h \in H^\infty(\Pi)$, *and* $\varphi_1 \in L_\infty(\mathbb{R})$. *If the operator* $\mathbb{H}(\varphi_1)$ *belongs* $\mathfrak{S}_1$, *then so does the operator* $\mathbb{H}(\varphi)$, *and*

$$\|\mathbb{H}(\varphi)\|_{\mathfrak{S}_1} \leq \|h\|_{L_\infty} \|\mathbb{H}(\varphi_1)\|_{\mathfrak{S}_1}$$

Remark

The symbol $\varphi_{j,x}(\lambda)$ *contains the multiplier* $T(\lambda) \in H^\infty(\Pi)$. *Therefore, in what follows, we consider the symbol*

$$\varphi^0_{j,x}(\lambda) = \lambda^j G_-(\lambda) e^{i\Phi(\lambda,x)} \tag{8}$$

Applying Pellier's Theorem to the Hankel operator with this symbol of the form, we must first estimate the integrals

$$I_j(\xi,x) := \frac{1}{2\pi i}\int_{-\infty}^{\infty}\left(\frac{1}{\tau-\xi}-\frac{\tau}{1+\tau^2}\right)\tau^j\overline{G_-(\tau)}e^{-i\Phi(\tau,x)}\,d\tau,$$
$$\xi\in\Pi,\ j=0,1,2,\ldots,$$

$$I_j^{(2)}(\xi,x) := \frac{1}{\pi i}\int_{-\infty}^{\infty}\frac{\tau^j\overline{G_-(\tau)}e^{-i\Phi(\tau,x)}}{(\tau-\xi)^3}\,d\tau,\quad \xi\in\Pi,\ j=0,1,2,\ldots. \tag{9}$$

The Saddle-Point Method

Using (4) we obtain

$$\overline{G_-(\tau)} = \int_0^\infty e^{i2\tau s} g(s)\, ds.$$

Here and further, we assume that $g(s) \geq 0$ almost everywhere and $g(s) \in L_1\left(\mathbb{R}_+, (1+s)^{j/2}\right)$. Thus, the integral (9) can be written as

$$I_j(\xi, x) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\tau - \xi} - \frac{\tau}{1+\tau^2} \right) \tau^j e^{-i\Phi(\tau,x)} \left(\int_0^\infty g(s) e^{i2\tau s} ds \right) d\tau. \tag{10}$$

Changing the order of integration, we obtain

$$I_j(\xi, x) = \frac{1}{2} \int_0^\infty g(s) J_j(s, \xi, x)\, ds,$$

where

$$J_j(s, \xi, x) := \frac{1}{\pi i} \int_{-\infty}^{\infty} \left(\frac{1}{\tau - \xi} - \frac{\tau}{1 + \tau^2} \right) \tau^j e^{-i\Phi(\tau, x-s)}\, d\tau,$$

$$\Phi(\tau, x - s) = 8t\tau^3 + 2(x - s)\tau.$$

Let us make the following change of variables

$$\tau = \beta(s)u, \quad \xi = \beta(s)\xi', \quad \text{where} \quad \beta(s) = \left(\frac{(s-x)}{12t}\right)^{1/2}.$$

Setting

$$S(u) = \frac{u^3}{3} - u, \quad \Lambda(s,x) := \Lambda(s) := \frac{(s-x)^{3/2}}{(3t)^{1/2}}$$

we obtain

$$J_j(s,\xi,x) := \widetilde{J}_j(s,\xi',x) = \beta^j(s)\widetilde{I}_j(s,\xi',x) - \beta^{j+2}(s)\widehat{I}_j(s,x),$$

where

$$\widetilde{I}_j(s,\xi',x) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u^j e^{-i\Lambda(s)S(u)}}{u-\xi'}\,du \qquad (11)$$

$$\widehat{I}(s,x) = \frac{1}{\pi i}\int_{-\infty}^{\infty} \frac{u^{j+1} e^{-i\Lambda(s)S(u)}}{1+\beta^2(s)u^2}\,du. \qquad (12)$$

Let us find a saddle-point contour for the integral (12). The critical points $u_\pm$ can be found from the equation

$$S'(u) = u^2 - 1 = 0, \quad u_\pm = \pm 1.$$

It is easy to calculate that

$$S(u_\pm) = \mp\frac{2}{3}, \quad S''(u_\pm) = \pm 2, \quad S'''(u_\pm) = 2.$$

Thus, the saddle-point contours are determined by the equations

$$S(u) + \frac{2}{3} = (u-1)^2 + \frac{1}{3}(u-1)^3 = -iv^2, \ v \in \mathbb{R}, \tag{13}$$

$$S(u) - \frac{2}{3} = -(u+1)^2 + \frac{1}{3}(u-1)^3 = -iv^2, \ v \in \mathbb{R}. \tag{14}$$

It is easy show that Esq. (13) and (14) are uniquely solvable for any $v \in \mathbb{R}$. We denote their solutions by $u_{\pm}(v)$ and introduce the saddle-point counters

$$\Gamma_{\pm} := \{z = u_{\pm}(v) | v \in \mathbb{R}\}.$$

It is easy to see that, in a neighborhood of the critical points $(u_{\pm}(0) = \pm 1)$, the following asymptotic relations hold:

$$\begin{aligned} u := \quad u_{+}(v) = \quad & 1 + e^{-i\frac{\pi}{4}} v + O(v^2), \quad v \in [-\varepsilon, \varepsilon] \\ u := \quad u_{-}(v) = \quad & -1 + e^{i\frac{\pi}{4}} v + O(v^2), \quad v \in [-\varepsilon, \varepsilon] \end{aligned}$$

Moreover, it is easy to see that, for sufficiently large v, we have

$$\left. \begin{array}{llll} u_{+}(v) \sim \sqrt[3]{3} & e^{i\frac{\pi}{2}} & |v|^{2/3}, & v \to -\infty \\ u_{+}(v) \sim \sqrt[3]{3} & e^{-i\frac{\pi}{6}} & v^{2/3}, & v \to +\infty \\ u_{-}(v) \sim \sqrt[3]{3} & e^{i\frac{\pi}{2}} & v^{2/3}, & v \to +\infty \\ u_{-}(v) \sim \sqrt[3]{3} & e^{i\frac{7}{6}\pi} & |v|^{2/3}, & v \to -\infty \end{array} \right\}$$

Estimation of the Second Term of Peller Theorem

Lemma

The integral (12) can be estimated as

$$|\widehat{I_j}(s,x)| \leq \frac{\mathrm{const}}{\beta^2(s)\Lambda^{1/2}(s)},$$

where "const" *is independent of* s *and* x.

Lemma

The integral (11) can be represented as

$$\widetilde{I}_j(s,\xi',x) = \widetilde{I_j^{+}}(s,\xi',x) + \widetilde{I_j^{-}}(s,\xi',x) + \widetilde{I}_{j,\mathrm{Res}}(s,\xi',x),$$

where

$$\left|\widetilde{I_j^{\pm}}(s,\xi',x)\right| \leq \mathrm{const} \begin{cases} \frac{1}{|\xi'\mp 1|\Lambda^{1/2}(s)}, & |\xi'\mp 1|\Lambda^{1/2}(s) \geq 1, \\ 1, & |\xi'\mp 1|\Lambda^{1/2}(s) \leq 1, \end{cases}$$

$$\left|\widetilde{I}^{0}_{j,\mathrm{Res}}(s,\xi',x)\right| \leq \mathrm{const},$$

and "const", *is independent of* s,ξ' *and* x.

Theorem

Let $I_j(\xi, x)$ *be the expression given by (10), and let* $g(s) \in L_1\left(\mathbb{R}_+, (1+s)^{j/2}\right)$. *Then, for* $j = 0, 1, \dots,$

$$|I_j(\xi, x)| \leq \begin{cases} c_1, & x \geq 0 \\ c_1 + c_2|x|^{j/2}, & x < 0, \end{cases}$$

where c_1 *and* c_2 *are independent of* ξ *and* x.

Substituting representation (4) into (9), we see that

$$I_j^{(2)}(\xi, x) = \frac{1}{\pi i} \int\limits_{-\infty}^{\infty} \frac{\tau^j e^{-i\Phi(\tau,x)}}{(\tau - \xi)^3} \left(\int\limits_0^{\infty} g(s) e^{i2\tau s} ds \right) d\tau. \qquad (15)$$

Changing the order of integration, we obtain the representation

$$I_j^{(2)}(\xi, x) = 2 \int\limits_{-\infty}^{\infty} g(s)\, J_j^{(2)}(s, \xi, x) ds,$$

where

$$J_j^{(2)}(s, \xi, x) := \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{\tau^j e^{-i\Phi(\tau,x-s)}}{(\tau - \xi)^3} d\tau. \qquad (16)$$

Making the same change of variables in the integral (16), we see that

$$J_j^{(2)}(s,\xi,x) = \beta(s)^{j-2}\widetilde{I_j^{(2)}}(s,\xi',x),$$

where

$$\widetilde{I_j^{(2)}}(s,\xi',x) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{u^j e^{-i\Lambda(s)S(u)}}{(u-\xi')^3}du. \tag{17}$$

Lemma

The integral (17) can be represented as

$$\widetilde{I_j^{(2)}}(s,\xi',x) = \widetilde{I_{j,+}^{(2)}}(s,\xi',x) + \widetilde{I_{j,-}^{(2)}}(s,\xi',x) + \widetilde{I_{j,Res}^{(2)}}(s,\xi',x),$$

where

$$|\widetilde{I_{j,\pm}^{(2)}}(s,\xi',x)| \le \mathrm{const} \begin{cases} \dfrac{1}{|\xi' \mp 1|^3 \Lambda^{1/2}(s)}, & |\xi' \mp 1|\Lambda^{1/2}(s) \ge 1 \\ \Lambda(s), & |\xi' \mp 1|\Lambda^{1/2}(s) \le 1 \end{cases}$$

$$|\widetilde{I}_{j,Res}^{(2)}(s,\xi',x)| \le \mathrm{const} \left\{ \Lambda(s)^{-\frac{j-2}{3}} |\xi''|^{j-2} \left(|\xi''|^6 + |\xi''|^3 + 1\right) e^{-c|\xi''|^3} \right\}$$

here $\xi'' = \xi' \Lambda^{1/3}(s), c > 0$ *and* $''\mathrm{const}''$ *are independent of* s, x *and* $\xi' \in \Pi \setminus (D_1 \cup D_{-1})$.

Theorem

Let $I_j^{(2)}(\xi', x)$ *be the function given by (15), and let* $g(s) \in L_1\left(\mathbb{R}_+, (1+|s|)^{j/2}\right)$. *Then*

$$A(x) := \int_{\Pi} |I_j^{(2)}(\xi, x)| d\xi \leq \begin{cases} c_3, \ x \geq 0 \\ c_3 + c_4|x|^{j/2}, \ x < 0, \end{cases}$$

where c_3 *and* c_4 *are independent of* x.

Applications to the Korteweg-de Vries Equation

$$\frac{\partial u(x,t)}{\partial t} - 6u(x,t)\frac{\partial u(x,t)}{\partial x} + \frac{\partial^3 u(x,t)}{\partial x^3} = 0, \quad t \geq 0, x \in \mathbb{R}.$$

$$u(x,0) = q(x),$$

$$\inf \operatorname{Spec}(\mathbb{L}_q) = -a^2 > -\infty \qquad \text{(is bounded below)}; \tag{18}$$

$$\int^{\infty} (1+|x|)^N |q(x)|\, dx < \infty, \quad N \geq 1 \quad \text{(decreases at } +\infty).$$

$$L_q = -\partial_x^2 + q \, - \, \text{Schrödinger operator.}$$

The condition

$$\sup_{|I|=1} \int_I \max(-q(x), 0)dx < \infty$$

is sufficient for (18).

Inverse Scattering Method (GGKM-Gardner, Green, Kruskal, Miuro)

1. Solving the Schrödinger equation $\mathbb{L}_q u = k^2 u$ we find $S_0 = \{R(k), (\kappa_n, c_n)\}$, where $R(k)$, $k \in \mathbb{R}$, is the reflection coefficient and (κ_n, c_n), $n = 1, 2, .., N$, are the so-called data on bound states associated with the eigenvalues, $-\kappa_n^2$.
2. $S(t) = \left\{ R(k) \exp\left(8ik^3 t\right),\ \kappa_n, c_n \exp\left(8\kappa_n^3 t\right) \right\}$.
3. Step 3 reduces to solving the inverse scattering problem for recovering the potential $u(x,t)$ (which now depends on $t \geq 0$) from $S(t)$. This procedure leads to the following explicit formula, which is usually called the *Dyson determinant*:

$$u(x,t) = -2\partial_x^2 \log\det\left(I + \mathbb{H}(x,t)\right).$$

Symbol of the Hankel Operator

$$\varphi_{x,t}(k) = R(k)\xi_{x,t}(k) + \int_0^a \frac{\xi_{x,t}(is)d\rho\,(s)}{s+ik},$$

where $-a^2$ is the lower bound of the spectrum of $\mathbb{L}_q$ and $\rho(s)$ is a measure with the properties

$$\operatorname{Supp}\rho \subseteq [0,a], \quad d\rho \geq 0, \quad \int_0^a d\rho < \infty.$$

$$\mathbb{H}\,(x,t) = \mathbb{H}(\Phi_{x,t}) + \mathbb{H}(\xi_{x,t}R_0),$$

where $\Phi_{x,t}$ is a meromorphic function in the upper half-plane (its particular form is inessential) and R_0 is the reflection coefficient of q bounded on $(0,\infty)$.

For R_0 we have the representation

$$R_0(\lambda) = T(\lambda) \int_0^\infty e^{-2i\lambda s} g(s)\, ds,$$

where $T \in H^\infty(\Pi)$, so that $T(\lambda) = O(1/\lambda)$, $|\lambda| \to \infty$, g is a function subject to the only constraint

$$|g(s)| \le |q(s)| + const \int_s^\infty |q|.$$

Global Classical Solution of KDV

1. $$\frac{\partial^{n+m}}{\partial x^n \partial t^m} \mathbb{H}\left(\Phi_{x,t}\right) \in \mathfrak{S}_1.$$
2. Main Theorem implies:

   For the operator $\mathbb{H}(\xi_{x,t} R_0)$, we proved that if

$$\int^{\infty} (1 + |s|)^N \, |q(s)| \, ds < \infty,$$

then

$$\frac{\partial^{n+m}}{\partial x^n \partial t^m} \mathbb{H}\left(\xi_{x,t} R_0\right) \in \mathfrak{S}_1$$

for all n and m, satisfying the condition

$$n + 3m \leq 2N - 1.$$

Theorem

Suppose that the (real) initial profile q satisfies the condition

$$\inf \operatorname{Spec}(\mathbb{L}_q) = -a^2 > -\infty \qquad \textit{(is bounded below)},$$

$$\int^{\infty} (1+|x|)^N \, |q(x)| \, dx < \infty, \quad N \geq 1 \quad (\textit{decreases} + \infty).$$

Then the function $\tau(x,t) := \det(1 + \mathbb{H}(x,t))$ *is well defined on* $\mathbb{R} \times \mathbb{R}_+$*, and its classical derivatives* $\partial^{n+m}\tau(x,t) \,/\, \partial x^n \partial t^m$ *exist provided that* $n + 3m \leq 2N - 1$*. Moreover, for* $N \geq 3$ *the Cauchy problem has a global (in time) classical solution which is given by*

$$u(x,t) = -2\frac{\partial^2}{\partial x^2} \log \tau(x,t), \quad t > 0.$$