Asymptotics of all eigenvalues of large Toeplitz matrices. From selfadjoint case to non selfadjoint

S. M. Grudsky

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Alexander Batalshchikov and Vladimir Stukopin.

Abstract

Analysis of the asymptotic behavior of the spectral characteristics of Toeplitz matrices as the dimension of the matrix tends to infinity has a history of over 100 years. For instance, quite a number of versions of Szegö's theorem on the asymptotic behavior of eigenvalues and of the so-called strong Szegö theorem on the asymptotic behavior of the determinants of Toeplitz matrices are known. Starting in the 1950s, the asymptotics of the maximum and minimum eigenvalues were actively investigated. However, investigation of the individual asymptotics of all the eigenvalues and eigenvectors of Toeplitz matrices started only quite recently: the first papers on this subject were published in 2009–2010.

It should be noted that the most of the results in this area are concerned to selfjoint case. These talk is devoted to some cases of the non selfjoint Toeplitz matrices.

Main object.

Spectral properties of larger finite Toeplitz matrices

$$A_n = (a_{j-k})_{j,k=0}^{n-1} = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \dots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \dots & a_{-(n-2)} \\ a_2 & a_1 & a_0 & \dots & a_{-(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-1} & a_{n-2} & a_{n-3} & \dots & a_0 \end{pmatrix}.$$

$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \; t \in \mathbb{T}$ -symbol of $\{A_n\}_{n=1}^{\infty}$

Eigenvalues, eigenvectors singular values, condition numbers, invertibility and norms of inverses, e.t.c.

$n \sim 1000$ is a business of numerical linear algebra.

Statistical physics - $n = 10^7 - 10^{12}$ - is a business of asymptotic theory.

Applications:

- the numerical solution of differential and integral equations;
- Ising model (in statistical mechanics);
- stochastic processes and time series analysis;
- signal processing;
- image processing;
- quantum mechanics.

Qualitative significance

Structure of location of eigenvalues

Distance between eigenvalues

Points of concentration

Fast numerical calculations

I. Two parameters:
n- dimensions of matrices;
j- number of eigenvalue

$$1 \leq j \leq n$$

Asymptotics by n uniformly in j.

II. Distance between λ_j and λ_{j+1} is small:

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n}\right) - \text{normal case}$$

$$|\lambda_j - \lambda_{j+1}| = O\left(\frac{1}{n^\gamma}\right) - \text{special case}$$

$$\lambda_j = \lambda_{j+1} \qquad\qquad - \text{exceptional case}$$

Publications about asymptotics of individual eigenvalues. Selfjoint case.

1. Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Inside the Eigenvalues of Certian of Hermitian Toeplitz Band Matrices. Computational and Applied Mathematics, 233 (2010), 2245-2264 pp.

2. Deift P, Its A, and Krasovsky I. Eigenvalues of Toeplitz matrices in the bulk of the spectrum. Bulletin of the Institute of Mathematics Academia Sinica (New Series) 7 (2012), 437-461 pp.

3. J. M. Bogoya, Albrecht Böttcher, Sergei M. Grudsky, Egor A. Maksimenko. Eigenvalues of Hermitian Toeplitz matrices with smooth simple-loop symbols. Journal of Mathematical Analysis and Applications Volume 422, Issue 2, 15 February 2015, 1308-1334 pp.

4. J.M. Bogoya, S.M. Grudsky and E.A. Maksimenko.. Eigenvalues of Hermitian Toeplitz Matrices Generated by Simple-loop Symbols with Relaxed Smoothness. Operator Theory: Advances and Applications, Volume 259, 2017, 179–212 pp.

Main results-selfjoint case

For $\alpha \geq 0$, we denote by W^α the weighted Wiener algebra of all functions $a : \mathbb{T} \to \mathbb{C}$ whose Fourier coefficients satisfy

$$\|a\|_\alpha := \sum_{j=-\infty}^{\infty} |a_j|(|j|+1)^\alpha < \infty.$$

Put $g(\varphi) := a(e^{i\varphi}),\ \varphi \in [0, 2\pi]$.

(I) a is real-valued;

(II) the range of g is a closed interval $[0, \mu]$ with $\mu > 0$, $g(0) = g(2\pi) = 0$, $g''(0) = g''(2\pi) > 0$, there is a $\varphi_0 \in (0, 2\pi)$ such that $g(\varphi_0) = \mu$, $g'(\sigma) > 0$ for $\sigma \in (0, \varphi_0)$, $g'(\sigma) < 0$ for $\sigma \in (\varphi_0, 2\pi)$, and $g''(\varphi_0) < 0$.

Symbols in the class SL^α are known as *simple-loop symbols*. (I) is equivalent to the condition that all matrices $T_n(a)$ ($n \in \mathbb{Z}_+$) are Hermitian (self-adjoint). If $a \in W^\alpha$, then $g \in C^{\lfloor \alpha \rfloor}[0, 2\pi]$ where $\lfloor \alpha \rfloor$ is the integer part of α. So, the condition $a \in \mathrm{SL}^\alpha$ with $\alpha \geq 1$ implies, in particular, that g belongs to $C^1[0, 2\pi]$.

In this work, for every $\alpha \geq 1$, we introduce a new class of symbols MSL^{α} (the modified simple loop class). Namely, $a \in \mathrm{MSL}^{\alpha}$ if $a \in \mathrm{SL}^{\alpha}$ and

(III) there exist functions $q_1, q_2 \in W^{\alpha}$ satisfying

$$a(t) = (t-1)q_1(t) \quad \text{and} \quad a(t) - a(e^{i\varphi_0}) = (t - e^{i\varphi_0})q_2(t). \qquad (1)$$

It is easy to proof that, if $a \in W^{\alpha}$, then q_1 and q_2 both belong to $W^{\alpha-1}$, but we require the stronger condition (III) instead.

Symmetric case

$$g(s) = g(2\pi - s).$$

Let further $\lambda = g(s)$

$$\begin{aligned}\beta(\sigma, s) :=& \frac{(g(\sigma) - g(s))e^{is}}{(e^{i\sigma} - e^{is})(e^{-i\sigma} - e^{is})} \\ =& \frac{g(s) - g(\sigma)}{4 \sin \frac{\sigma - s}{2} \sin \frac{\sigma + s}{2}}.\end{aligned}$$

We will show that β is a continuous and positive function on $[0, 2\pi] \times [0, \pi]$. We define the function $\eta : [0, \pi] \to \mathbb{R}$ by

$$\eta(s) := \theta(\psi(s)) = \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma - s}{2}} \, d\sigma - \frac{1}{4\pi} \int_0^{2\pi} \frac{\log \beta(\sigma, s)}{\tan \frac{\sigma + s}{2}} \, d\sigma,$$

the integrals taken in the principal-value sense.

Theorem (3)

Let $\alpha \geq 1$ *and* $a \in \mathrm{MSL}^{\alpha}$. *Then for every* $n \geq 1$:

(I) *the eigenvalues of* $T_n(a)$ *are all distinct:* $\lambda_1^{(n)} < \cdots < \lambda_n^{(n)}$;

(II) *the numbers* $s_j^{(n)}$, *such that* $\lambda_j^{(n)} = g(s_j^n)$ $(j = 1, \ldots, n)$ *satisfy*

$$(n+1)s_j^{(n)} + \eta(s_j^{(n)}) = \pi j + \Delta_1^{(n)}(j), \tag{2}$$

where $\Delta_1^{(n)}(j) = o\left(\frac{1}{n^{\alpha-1}}\right)$ *as* $n \to \infty$, *uniformly in* j;

(III) *for every sufficiently large* n, (2) *has exactly one solution* $s_j^{(n)} \in [0, \pi]$ *for each* $j = 1, \ldots, n$.

Theorem (4)

Let $d_j^{(n)} = \dfrac{\pi j}{n+1}$, *then under the conditions of previous Theorem*

$$\lambda_j^{(n)} = g(d_j^{(n)}) + \sum_{k=1}^{\lfloor\alpha\rfloor} \frac{r_k(d_j^{(n)})}{(n+1)^k} + \Delta_3^{(n)}(j),$$

where $\Delta_3^{(n)}$ *is* $o\Big(\frac{1}{n^\alpha}(d_j^{(n)}(\pi - d_j^{(n)}))^{\alpha-1}\Big)$ *if* $1 \le \alpha < 2$ *and* $o\Big(\frac{d_j^{(n)}}{n^\alpha}(\pi - d_j^{(n)})\Big)$ *if* $\alpha \ge 2$ *as* $n \to \infty$, *uniformly in* j. *The coefficients* r_k *can be calculated explicitly; in particular,*

$$r_1(s) = -g'(s)\eta(s) \quad \text{and} \quad r_2(s) = \frac{1}{2}g''(s)\eta^2(s) + g'(s)\eta(s)\eta'(s).$$

Symmetric complexvalue symbol

$$a(t) = \sum_{j=-\infty}^{\infty} a_j t^j, \; a_j = a_{-j}.$$

$$a_1(t) = c_1 \sin(c_0 t^2) + \frac{1}{20}\left((1+t)^{5/2} + (1-t)^{5/2}\right), \; t \in [-\pi; \pi],$$

$$c_0 = \frac{1}{5} - \frac{1}{6} i, \; c_1 = \frac{(1-\pi)^{3/2} - (\pi+1)^{3/2}}{16\pi c_0 \cos(\pi^2 c_0)}$$

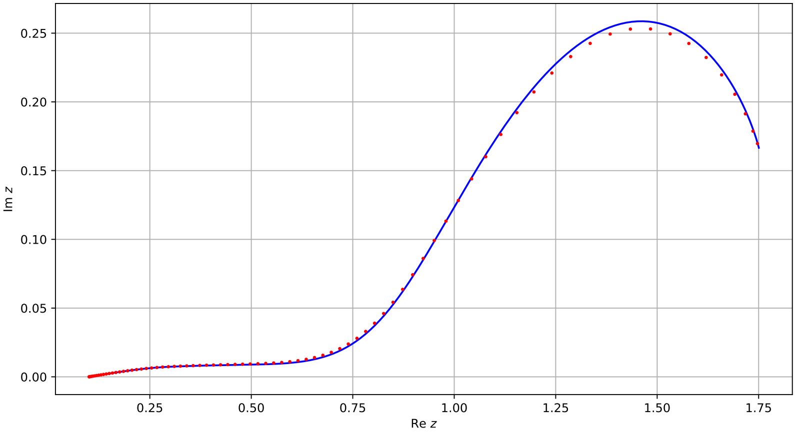

Figure: Image of the symbol $a_1(t)$ and eigenvalues of the matrix $T_{80}(a_1)$

$$a(t) \in W^{\alpha},\ \alpha \geq 2,\ g(\varphi) = a\left(e^{i\varphi)}\right),\ \varphi \in [0, 2\pi].$$

$$\|a\|_{\alpha} := \sum_{j=-\infty}^{\infty} |a_j| \left(|j| + 1\right)^{\alpha} < \infty.$$

I. $g(\varphi) = g(2\pi - \varphi)$ – symmetric.

II. $\mathcal{R}(a)$ – is simple curve without self–intersection
$\mathcal{R}(a) = (M_0, M_1),\ M_0 \neq M_1.$

III. $g'(\varphi) \neq 0,\ \varphi \in (0, \pi).$

IV. $g''(0) = g''(2\pi) \neq 0,\ g''(\pi) \neq 0.$

Main Idea

$$T_n(a) = T_n(a_n) \quad \textbf{!}$$

$$a_n(t) = \sum_{j=-(n-1)}^{n-1} a_j t^j, \; g_n(\psi) = a_n\left(e^{i\psi}\right),$$

where

$$\psi \in \Pi_n := \left\{\psi = \varphi + i\delta | \varphi \in [c/n, \; \pi - c/n], \; \delta \in [-C/n, \; C/n]\right\}$$

with $c > 0$ and $C > 0$.

$$\mathcal{R}_n(a) =: \left\{g_n(\psi) : \psi \in \Pi_n\right\}$$

Lemma (1)

Let $a(t) \in W^\alpha$, $\alpha \geq 2$, *and satisfies condition 1.-2.-3.-4. Then the map*

$$g_n(\psi) : \Pi_n \to \mathcal{R}_n(a) \text{ is bijection for } n \text{ large enough.}$$

Lemma (2)

Let $\psi = (\varphi + i\delta) \in \Pi_n$, $a(t) \in W^\alpha$, $\alpha \geq 0$, *and* $m = [\alpha]$. *Then*

$$g_n(\psi) = g(\varphi) + \sum_{k=1}^{m} \frac{g^{(k)}(\varphi)}{k!}(i\delta)^k + \sum_{k=0}^{m+1} \alpha_{n,k}(\varphi)(i\delta)^k,$$

where $\alpha_{n,k}(\varphi) \in W^0$, *and*

$$\|\alpha_{n,k}\|_0 = o(n^{k-\alpha}),\ k = 0, 1, \ldots, m$$

and

$$\|\alpha_{n,m+1}\|_0 = O(n^{m+1-\alpha}).$$

Let $g_n(\varphi_{1,n}(\lambda)) = \lambda$, then introduce following functions

$$\hat{b}_n(t,\lambda) = \frac{(a_n(t)-\lambda)e^{i\varphi_{1,n}(\lambda)}}{\left(t - e^{i\varphi_{1,n}(\lambda)}\right)\left(t^{-1} - e^{i\varphi_{1,n}(\lambda)}\right)},\ \lambda \in \mathcal{R}_n(a),$$

$$\theta_n(\lambda) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log \hat{b}_n(\tau,\lambda)}{\tau - e^{i\varphi_{1,n}(\lambda)}} d\tau - \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log \hat{b}_n(\tau,\lambda)}{\tau - e^{-i\varphi_{1,n}(\lambda)}} d\tau,\ \lambda \in \mathcal{R}_n(a)$$

$$\eta_n(s) := \theta_n(g_n(s)),\ s \in \Pi_n.$$

Consider two sequences

$$d_{j,n} = \frac{\pi j}{n+1},\ e_{j,n} = d_{j,n} - \frac{\eta_n(d_{j,n})}{n+1}.$$

Introduce small domains

$$\Pi_{j,n}(a) := \left\{ s \in \Pi_n(a),\ |s - e_{j,n}| \le \frac{c_n}{n+1} \right\},$$

where $c_n \to 0$.

Theorem (5)

Let $a \in \mathrm{CSL}^\alpha$, $\alpha \geq 2$. *Then for sufficiently large natural number* n *the following statements hold:*

1. *all eigenvalues* $T_n(a)$ *are different, and* $\lambda_{j,n} \in g(\Pi_{j,n}(a))$ *for* $j = 1,\ 2,\ \ldots,\ n$
2. *values* $s_{j,n}$ *such that* $\lambda_{j,n} = g_n(s_{j,n})$ *satisfy the equation*

$$(n+1)s + \eta_n(s) = \pi j + \Delta_n(s), \qquad j = 1,\ 2,\ \ldots,\ n \tag{3}$$

   with $|\Delta_n(s)| = o(1/n^{\alpha-2})$ *where* $\Delta_n(s) \to 0$ *as* $n \to \infty$ *uniformly respect to* $s \in \Pi_n(a)$.
3. *Equation (3) have a unique solution in the domain* $\Pi_{j,n}(a)$.

Theorem (6)

Under conditions of Theorem 5

$$\lambda_j^{(n)} = g(d_{j,n}) + \sum_{k=1}^{[\alpha]-1} \frac{r_k(d_{j,n})}{(n+1)^k} + \Delta_3^{(n)}(j) \tag{4}$$

where

$$\Delta_3^{(n)}(j) = \begin{cases} o\left(\frac{d(\pi-d)}{n}\right), & \alpha = 2, \\ O\left(\frac{d(\pi-d)}{n^{\alpha-1}}\right), & \alpha > 2. \end{cases}$$

as $n \to \infty$ *uniformly in* j *and* $d = d_{j,n}$. *The coefficients* r_k *can be calculated explicitly; in particular*

$$r_1(\varphi) = -g'(\varphi)\eta(\varphi) \quad \textit{and} \quad r_2(\varphi) = \frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi).$$

Remark

For small j, $\left(j^2/n \to 0\right)$ *we have following asymptotics from Theorem 6, for* $\alpha \geq 2$,

$$\lambda_{j,n} = g(0) + C_3 \frac{j^2}{(n+1)^2} + o\left(\frac{j^2}{(n+1)^2}\right),$$

where

$$C_3 = \frac{\pi^2 g''(0)}{2}.$$

Location the eigenvalue relative to $\mathcal{R}(a)$

$$\lambda_{j,n} = g(d_{j,n}) - \frac{g'(d_{j,n})\,\eta(d_{j,n})}{n+1} + O\left(\frac{1}{n^2}\right).$$

Let $\widetilde{e_{j,n}} = d_{j,n} + \dfrac{\operatorname{Re}\eta(d_{j,n})}{n+1}$, then

$$\lambda_{j,n} = g(\widetilde{e_{j,n}}) + i\frac{g'(\widetilde{e_{j,n}})\,(\operatorname{Im}\eta(d_{j,n}))}{n+1} + O\left(\frac{1}{n^2}\right).$$

$\lambda_{j,n}$ is located on the normal to curve $\mathcal{R}(a)$ in the point $z = g(\widetilde{e_{j,n}})$ with exactitude to $O\left(\frac{1}{n^2}\right)$

$$\frac{|g'(e_{j,n})\operatorname{Im}\eta(d_{j,n})|}{n+1} + O\left(\frac{1}{n^2}\right) - \text{distance between } \lambda_{j,n} \text{ and } \mathcal{R}(a).$$

Numerical example

Approximations:

$$\lambda_{1,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1},$$

$$\lambda_{2,j}^{(n)} = g(d_{j,n}) - \frac{g'(\varphi)\eta(\varphi)}{n+1} + \frac{\frac{1}{2}g''(\varphi)\eta^2(\varphi) + g'(\varphi)\eta(\varphi)\eta'(\varphi)}{(n+1)^2}.$$

Errors:

$$\Delta_1^{(n)} = \max_j \left| \frac{\lambda_{1,j} - \lambda_j}{\lambda_j} \right|,$$

$$\Delta_2^{(n)} = \max_j \left| \frac{\lambda_{2,j} - \lambda_j}{\lambda_j} \right|,$$

$$a_1(t) \in W^{2,5-\delta}, \ \forall \quad \delta > 0$$

n	20	40	80	160	320
$\Delta_1^{(n)}$	3.2e-03	8.8e-04	2.3e-04	5.9e-05	1.5e-05
$\Delta_2^{(n)}$	3.9e-04	5.6e-05	7.2e-06	9.2e-07	1.2e-07