Uniform Boundedness of Toeplitz Matrices with Variable Coefficients.

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Let T be the complex unit circle and $a : [0,1] \times [0,1] \times T \to C$ be a continuous function. We formally represent a by its Fourier series in the last variable,

$$a(x,y,t) = \sum_{n=-\infty}^{\infty} \hat{a}_n(x,y)t^n, \quad \hat{a}_n(x,y) = \int_T a(x,y,t)t^{-n}\frac{|dt|}{2\pi}.$$

The $(N+1) \times (N+1)$ variable-coefficient Toeplitz matrix generated by a is the matrix

$$A_N(a) = \left(\hat{a}_{j-k}\left(\frac{j}{N}, \frac{k}{N}\right)\right)_{j,k=0}^{N}.$$

This report concerned with weakest conditions on a that guarantee the uniform boundedness of the spectral norms $||A_N(a)||_\infty$ as $N \to \infty$. It is easily seen that

$$||A_N(a)||_\infty \leq \sum_{n=-\infty}^{\infty} M_{\infty,\infty}(\hat{a}_n) := \sum_{n=-\infty}^{\infty} \sup_{x\in[0,1]} \sup_{y\in[0,1]} |\hat{a}_n(x,y)|.$$

Hence, $\sup ||A_N(a)||_\infty < \infty$ whenever a is subject to the Wiener type condition

$$\sum_{n=-\infty}^{\infty} M_{\infty,\infty}(\hat{a}_n) < \infty.$$

M. Kac, W. L. Murdock and G. Szegö 1953, Simonenko I.B. 2000-2005,
Erhard T. and Chao B. 2001-2004.

If a does not depend on the first two variables,

$$a(t) = \sum_{n=-\infty}^{\infty} \hat{a}_n t^n, \quad \hat{a}_n = \int_T a(t) t^{-n} \frac{|dt|}{2\pi},$$

then $A_N(a)$ is the pure Toeplitz matrix $T_N(a) := (\hat{a}_{j-k})_{j,k=0}^{N}$ and the above inequality for $||A_N(a)||_\infty$ amounts to the inequality

$$||T_N(a)||_\infty \le \sum_{n=-\infty}^{\infty} |\hat{a}_n| =: ||a||_W.$$

It is well known that actually

$$||T_N(a)||_\infty \le M_\infty(a) := \sup_{t\in T} |a(t)|$$

and that this is even true if a is an arbitrary function in $L^\infty(T)$;
The bound $||a||_W$ is much weaker than the bound $M_\infty(a)$, and this leads to the question whether there is a substitute for the bound $\sum_{n=-\infty}^{\infty} M_{\infty,\infty}(\hat{a}_n)$ of the type $||T_N(a)||_\infty \le M_\infty(a)$.

Question 1. Is there $a \in C([0,1] \times [0,1] \times T)$ such that

$$\sup_N ||A_N(a)||_\infty = \infty?$$

Answer: Yes!

Question 2. Whether $\sup ||A_N(a)||_\infty$ is finite if a has some smoothness in the first two variables. While in third variable $a(x, y, \cdot) \in L_\infty(T)$?

Counterexamples

Theorem

There exist functions $a(x,t)$ in $C([0,1] \times T)$ such that

$$\sup_{N \geq 0} ||A_N(a)||_\infty = \infty.$$

Proof. Assume the contrary, that is, $\sup ||A_N(a)||_\infty < \infty$ for every function a in $C([0,1] \times T)$. Let S denote the Banach space of all sequences $\{B_N\}_{N=0}^\infty$ of matrices $B_N \in C^{(N+1)\times(N+1)}$ such that

$$||\{B_N\}_{N=0}^\infty|| := \sup_{N \geq 0} ||B_N||_\infty < \infty.$$

By our assumption, the map

$$T : C([0,1] \times T) \to S, \quad a \mapsto \{A_N(a)\}_{N=0}^{\infty}$$

is a linear operator defined on all of $C([0,1] \times T)$. T is bounded. (According to the closed graph theorem.) It means that there is a const $C < \infty$ such that

$$||A_N(a)||_\infty \leq C M_{\infty,\infty}(a)$$

for all $a \in C([0,1] \times T)$.

Fix $N \geq 2$ and for $j = 1, \ldots, N-1$, denote by I_j the segment

$$I_j = \left[\frac{j}{N} - \frac{1}{2N}, \frac{j}{N} + \frac{1}{2N} \right].$$

Let a_j be the function that is identically zero on $[0,1] \backslash I_j$, increases linearly from 0 to 1 on the left half of I_j, and decreases linearly from 1 to 0 on the right half of I_j. Put

$$a(x,t) = a_1(x)t^1 + a_2(x)t^2 + \ldots + a_{N-1}(x)t^{N-1}.$$

As the spectral norm of a matrix is greater than or equal to the ℓ^2 norm of its first column and as $\hat{a}_j(t) = a_j(x)$ for $1 \le j \le N-1$, it follows that

$$||A_N(a)||_\infty^2 \ge \sum_{j=1}^{N-1} \left| a_j\left(\frac{j}{N}\right) \right|^2 = \sum_{j=1}^{N-1} 1^2 = N-1. \qquad (!)$$

Since $a(x,t) = 0$ for $x \notin \cup I_j$ and $|a(x,t)| = |a_j(x)t^j| \le 1$ for $x \in I_j$, we obtain that $M_{\infty,\infty}^2(a) = 1$. Consequently, (!) gives $N - 1 \le C^2 \cdot 1$ for all $N \ge 2$, which is impossible.

Definition 1.

Let $0 < \alpha \leq 1$. We say that a continuous functions $a(x, t)$ on $[0, 1] \times T$ is in $H_{\alpha,\infty}$
if

$$M_{\alpha,\infty}(a) := \sup_{t \in T} \sup_{x_1, x_2} \frac{|a(x_2, t) - a(x_1, t)|}{|x_2 - x_1|^{\alpha}} < \infty,$$

Theorem

If $0 < a < 1/2$, there exist functions $a(x, t)$ in $H_{\alpha,\infty}$ such that

$$\sup_{N \geq 0} ||A_N(a)||_{\infty} = \infty.$$

Sufficient conditions.

Case of symbols $a(x,t)$.

$$M_{\infty,\infty}(a) = \sup_{x\in[0,1]} \sup_{t\in T} |a(x,t)|$$

$$M_{1+\alpha,\infty}(a) := M_{\alpha,\infty}\left(\frac{\partial a}{\partial x}\right)$$

Theorem (A)

Let $\alpha > 0$. There exists a constant $C(\alpha)$ depending only on α such that

$$||A_N(a)||_\infty \leq C(\alpha)(M_{\infty,\infty}(a) + M_{1+\alpha,\infty}(a))$$

for all functions $a(x,t)$ in $H_{1+\alpha,\infty}$.

Lemma

If $f(x)$ is a function in $H_{1+\alpha}$ and $f(0) = f(1)$, then

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{2\pi i n x}.$$

with

$$|f_n| \leq \frac{M_\alpha(f')}{2^{2+\alpha}\pi|n|^{1+\alpha}} \text{ for } |n| \geq 1.$$

Proof of the Theorem (A)

Proof. We write $a = a_0 + a_1$ with

$$a_1(x,t) = (a(1,t) - a(0,t))x + a(0,t), \quad a_0(x,t) = a(x,t) - a_1(x,t).$$

Then $A_N(a) = A_N(a_0) + A_N(a_1)$. Obviously,

$$A_N(b(x)c(t)) = \left(b\left(\frac{j}{N}\right) \hat{c}_{j-k} \right)_{j,k=0}^{N} = D_N(b)T_N(c), \qquad (!!)$$

where $D_N(b) = \mathrm{diag}(b(j/N))_{j=0}^{N}$ and $T_N(c) = (\hat{c}_{j-k})_{j-k=0}^{N}$.
Taking into account that $||D_N(b)||_\infty \le M_\infty(b)$ and
$||T_N(c)||_\infty \le M_\infty(c)$, we obtain that

$$||A_N(a_1)||_\infty \le M_\infty(x)M_\infty(a(1,t) - a(0,t)) + M_\infty(a(0,t)) \le 3M_{\infty,\infty}(a).$$

As $a_0(0,t) = a_0(1,t)(= 0)$, Lemma gives

$$a_0(x,t) = \sum_{n=-\infty}^{\infty} a_n^0(t) e^{2\pi i n x}$$

with

$$|a_n^0(t)| \le \frac{M_\alpha(\partial_x a_0(x,t))}{2^{2+\alpha}\pi |n|^{1+\alpha}} \qquad (!!!)$$

for $|n| \ge 1$. From (!!) we infer that

$$||A_N(a_n^0(t)e^{2\pi i n x})||_\infty \le M_\infty(e^{2\pi i n x}) M_\infty(a_n^0(t)) = M_\infty(a_n^0).$$

Thus, by (!!!),

$$\begin{aligned} ||A_N(a_0||_\infty &\leq M_\infty(a_0^0) + \sum_{|n|\geq 1} M_\infty(a_n^0) \\ &\leq M_\infty(a_0^0 + \frac{1}{2^{2+\alpha}\pi} \sum_{|n|\geq 1} \frac{M_{\alpha,\infty}(\partial_x a_0(x,t))}{|n|^{1+\alpha}}. \end{aligned}$$

Since $a_0(x,t) = a(x,t) - a_1(x,t)$ and $\partial_x a_1(x,t)$ is independent of x, we get

$$M_{\alpha,\infty}(\partial_x a_0(x,t)) = M_{\alpha,\infty}(\partial_x a(x,t)) = M_{1+\alpha,\infty}(a).$$

Furthermore,

$$
\begin{aligned}
M_\infty(a_0^0) &= \sup_{t\in T}\left|\int_0^1 a_0(x,t)dx\right| \le M_{\infty,\infty}(a_0) = M_{\infty,\infty}(a-a_1)\\
&\le M_{\infty,\infty}(a) + M_{\infty,\infty}(a_1) \le 4M_{\infty,\infty}(a).
\end{aligned}
$$

In summary,

$$
||A_N(a)||_\infty \le 7M_{\infty,\infty}(a) + \left(\frac{1}{2^{2+\alpha}\pi}\sum_{|n|\ge 1}\frac{1}{|n|^{1+\alpha}}\right) M_{1+\alpha,\infty}(a),
$$

which implies the assertion at once.

$a(x,t) \in H_{\alpha,\infty}(a)$

$$\sup_{N\geq 0} ||A_N(a)||_\infty < \infty?$$

If $\alpha > 1 \Rightarrow$ yes!

If $\alpha < \frac{1}{2} \Rightarrow$ no

$$\alpha \in [\frac{1}{2}; 1.]?$$

Theorem (B)

If $a(x,t)$ is a function in $H_{\alpha,\infty}$ with $\alpha > 1/2$, then there is a constant $C(\alpha) < \infty$ depending only on α such that

$$||A_N(a)||_\infty \leq C(\alpha)(M_{\infty,\infty}(a) + M_{\alpha,\infty}(a)) \quad \text{for all} \quad N \geq 0.$$

$$\alpha = \frac{1}{2}?$$

Case of the symbol $a(x, y, t)$

Let $0 < \alpha \leq 1$. We denote by $H_{\alpha,\alpha,\infty}$ the set of all continuous functions a: $[0,1] \times [0,1] \times T \to C$ for which

$$M_{\alpha,\infty,\infty}(a) := \sup_{t\in T} \sup_{y\in[0,1]} \sup_{x_1,x_2} \frac{|a(x_2,y,t) - a(x_1,y,t)|}{|x_2 - x_1|^\alpha} < \infty,$$

$$M_{\infty,\alpha,\infty}(a) := \sup_{t\in T} \sup_{x\in[0,1]} \sup_{y_1,y_2} \frac{|a(x,y_2,t) - a(x,y_1,t)|}{|y_2 - y_1|^\alpha} < \infty,$$

and

$$M_{\alpha,\alpha,\infty}(a) := \sup_{t\in T} \sup_{x_1,x_2} \sup_{y_1,y_2} \frac{|\Delta_2 a(x_1,x_2,y_1,y_2,t)|}{|x_2 - x_1|^\alpha |y_2 - y_1|^\alpha} < \infty$$

where $\Delta_2 a(x_1, x_2, y_1, y_2, t)$ is the second difference

$$\Delta_2 a(x_1, x_2, y_1, y_2, t) = a(x_2, y_2, t) - a(x_2, y_1, t) - (a(x_1, y_2, t) - a(x_1, y_1, t))$$

and $\sup_{z_1, z_2}$ means the supremum over all $z_1, z_2 \in [0, 1]$ such that $z_1 \neq z_2$.

Theorem

Let $a(x, y, t)$ be a function in $H_{\alpha,\alpha,\infty}$ with $\alpha > 1/2$. Then there exists a constant $E(\alpha) < \infty$ depending only on α such that

$$||A_N(a)||_\infty \leq E(\alpha)(M_{\alpha,\alpha,\infty}(a) + M_{\alpha,\infty,\infty}(a) + M_{\infty,\alpha,\infty}(a) + M_{\infty,\infty,\infty}(a))$$

for all $N \geq 0$.

Discontinuous generating functions

For $0 < \alpha \leq 1$, we denote by $H_{\alpha,\alpha}$ the Banach space of all continuous functions $f : [0,1]^2 \to C$ for which

$$||f||_\alpha := M_{\infty,\infty}(f) + M_{\alpha,\infty}(f) + M_{\infty,\alpha}(f) + M_{\alpha,\alpha}(f) < \infty,$$

where $M_{\infty,\infty}(f)$ is the maximum of $|f(x,y)|$ on $[0,1]^2$ and

$$M_{\alpha,\infty}(f) = \sup_{y\in[0,1]} \sup_{x_1,x_2} \frac{|f(x_2,y) - f(x_1,y)|}{|x_2 - x_1|^\alpha},$$

$$M_{\infty,\alpha}(f) = \sup_{x\in[0,1]} \sup_{y_1,y_2} \frac{|f(x,y_2) - f(x,y_1)|}{|y_2 - y_1|^\alpha},$$

$$M_{\alpha,\alpha}(f) = \sup_{x_1,x_2} \sup_{y_1,y_2} \frac{|\Delta_2 f(x_1,x_2,y_1,y_2)|}{|x_2 - x_1|^\alpha |y_2 - y_1|^\alpha},$$

Let $L^\infty(T, H_{\alpha,\alpha})$ be the set of all measurable and essentially bounded functions $a : T \to H_{\alpha,\alpha}$.

Theorem

Let $a \in L^\infty(T, H_{\alpha,\alpha})$*, where* $\alpha > 1/2$*. Then*

$$||A_N(a)||_\infty \le D(\alpha) \sup_{t\in T} |a(x,y,t)||_{\alpha,\alpha}$$

with some constant $D(\alpha) < \infty$ *depending only on* α*.*

Theorem

Let $a \in L^\infty(T, C^2([0,1] \times [0,1])$*, then*

$$\sup_N ||A_N(a)|| < \infty.$$