Bol. Soc. Mat. Mexicana (2) Vol. 10, 2004

DYNAMICS OF PROPERTIES OF TOEPLITZ OPERATORS ON THE UPPER HALF-PLANE: HYPERBOLIC CASE

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ABSTRACT. We consider Toeplitz operators $T_a^{(\lambda)}$ acting on the weighted Bergman spaces $\mathcal{A}_\lambda^2(\Pi)$, $\lambda \in [0, \infty)$, over the upper half-plane Π, whose symbols depend on $\theta = \arg z$. Motivated by the Berezin quantization procedure we study the dependence of the properties of such operators on the parameter of the weight λ and, in particular, under the limit $\lambda \to \infty$.

1. Introduction

This is a part of the two-paper set devoted to the study of Toeplitz operators acting on weighted Bergman spaces on the upper half-plane. Both are motivated by the same ideas and are a continuation of our research started in [6]. We have mentioned in [6] the papers [1, 2, 3, 9, 10], where Toeplitz operators with smooth (or continuous) symbols acting on the weighted Bergman spaces, as well as C^*-algebras generated by such operators, appear naturally in the context of problems in mathematical physics. In particular, recall that given a smooth symbol $a = a(z)$, the family of Toeplitz operators $T_a = \{T_a^{(h)}\}$, with $h \in (0, 1)$, is considered under the Berezin quantization procedure [1, 2]. For a fixed h the Toeplitz operator $T_a^{(h)}$ acts on the weighted Bergman space $\mathcal{A}_h^2$. In the special quantization procedure each Toeplitz operator $T_a^{(h)}$ is represented by its Wick symbol $\widetilde{a}_h$, and the correspondence principle says that for smooth symbols one has

$$\lim_{h \to 0} \widetilde{a}_h = a.$$

Moreover by [8] the above limit remains valid in the L_1-sense for a wider class of symbols.

The same, as in a quantization procedure, weighted Bergman spaces appear naturally in many questions in complex analysis and operator theory. In the last cases a weight parameter is normally denoted by λ and runs through $(-1, +\infty)$. In the sequel we will consider weighted Bergman spaces $\mathcal{A}_\lambda^2$ parameterized by $\lambda \in (-1, +\infty)$ which is connected with $h \in (0, 1)$, used as the parameter in the quantization procedure, by the rule $\lambda + 2 = \frac{1}{h}$.

At this stage an important problem emerges: *study of the behavior of different properties (boundedness, compactness, spectral properties, etc.)* of $T_a^{(\lambda)}$

2000 *Mathematics Subject Classification*: 47B35.

Keywords and phrases: Toeplitz operators, weighted Bergman spaces, boundedness, spectrum.

Research supported by CONACYT Grant U40654-F.

The first author supported by CONACYT Grant, Cátedra Patrimonial, Nivel II, No 010186.

in dependence on λ, and comparison of their limit behavior under $\lambda \to \infty$ with corresponding properties of the initial symbol a.

It seems to be quite impossible to get a reasonably complete answer to the above problem for general (smooth) symbols, even for the simplest case of the weighted Bergman spaces on the unit disk (hyperbolic plane). At the same time the recently discovered classes of commutative *-algebras of Toeplitz operators on the unit disk suggest classes of symbols for which a satisfactory complete answer can be given. Recall in this connection (for details see [11, 12]) that all known cases of commutative *-algebras of Toeplitz operators on the unit disk are classified by pencils of (hyperbolic) geodesics of the following three possible types: geodesics intersecting in a single point (elliptic pencil), parallel geodesics (parabolic pencil), and disjoint geodesics, i.e., all geodesics orthogonal to a given one (hyperbolic pencil). Symbols which are constant on the cycles, the orthogonal trajectories to the geodesics forming a pencil, generate in each case a commutative *-algebra of Toeplitz operators. Moreover these commutative properties of Toeplitz operators do not depend at all on smoothness properties of symbols, the symbols can be merely measurable.

The model case for elliptic pencils, Toeplitz operators on the unit disk with radial symbols, was considered in [6]. In the present paper we consider the model case for hyperbolic pencils, while another paper [5] of this two-paper set is devoted to the study of the model case for parabolic pencils. Both papers together cover the part remaining after [6]. The results for other (non model) cases can be easily obtained by means of Möbius transformations.

We study Toeplitz operators on the upper half-plane equipped with the hyperbolic metric, where the model case for hyperbolic pencils is realized as Toeplitz operators with symbols depending only on $\theta = \arg z$.

The key feature of symbols constant on cycles, which permits us to obtain much more complete information than when studying general symbols, is as follows. In each case of a commutative *-algebra generated by Toeplitz operators the Toeplitz operators admit a spectral type representation, i.e., they are unitary equivalent to multiplication operators, by a certain sequence in the elliptic case and by certain functions on $\mathbb{R}_+$ and $\mathbb{R}$ in the parabolic and hyperbolic cases, respectively.

We mention a difference between the previously studied elliptic case [6] and the remaining cases. In particular, in the elliptic case the Toeplitz operators have a discrete spectrum and can be compact even having symbols unbounded near the boundary, while in both the parabolic and hyperbolic cases the Toeplitz operators always have only a continuous spectrum and, being nonzero, can not be compact.

As in the preceding paper [6], the word "dynamics" in the title stands for the emphasis on our main theme: *what happens to properties of Toeplitz operators acting on weighted Bergman spaces when the weight parameter varies.*

In the paper, as is a custom in operator theory, we consider weighted Bergman spaces depending on a real parameter $\lambda \in (-1, \infty)$.

Denote by Π the upper half-plane in $\mathbb{C}$, and introduce the weighted Hilbert space $L_2(\Pi, d\mu_\lambda)$ which consists of measurable functions f on Π for which the norm

$$\|f\|_{L_2(\Pi,d\mu_\lambda)} = \left(\int_\Pi |f(z)|^2 d\mu_\lambda(z) \right)^{1/2}$$

is finite. Here $d\mu_\lambda(z) = \mu_\lambda(z)dv(z)$ with

$$\mu_\lambda(z) = (\lambda+1)(2\mathrm{Im}\, z)^\lambda, \quad dv(z) = \frac{1}{\pi}\, dxdy, \ z = x+iy.$$

Let further $\mathcal{A}^2_\lambda(\Pi)$ denote the weighted Bergman space defined to consist of functions belonging to $L_2(\Pi, d\mu_\lambda)$ and analytic in the upper half-plane Π.

It is well known (see, for example, [10]) that the orthogonal Bergman projection $B_{\Pi,\lambda}$ of $L_2(\Pi, d\mu_\lambda)$ onto the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi)$ has the form

$$\begin{aligned}(B_{\Pi,\lambda}f)(z) &= (\lambda+1)\int_\Pi f(\zeta)\left(\frac{\zeta-\overline{\zeta}}{z-\overline{\zeta}}\right)^{\lambda+2}\frac{dv(\zeta)}{(2\,\mathrm{Im}\,\zeta)^2} \\ &= i^{\lambda+2}\int_\Pi \frac{f(\zeta)}{(z-\overline{\zeta})^{\lambda+2}}\, d\mu_\lambda(\zeta).\end{aligned}$$

Given a function (symbol) $a = a(z)$, $z \in \Pi$, the Toeplitz operators $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ is defined as follows

$$T_a^{(\lambda)} f = B_{\Pi,\lambda} af, \qquad f \in \mathcal{A}^2_\lambda(\Pi).$$

The key result, which gives an easy access to the properties of Toeplitz operators studied in the paper, is established in Section 2. Namely, we prove that the Toeplitz operator $T_a^{(\lambda)}$ with symbol $a(\theta)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$ acting on $L_2(\mathbb{R})$, where

$$\gamma_{a,\lambda}(\xi) = 2^\lambda\ (\lambda+1)\vartheta^2_\lambda(\xi)\int_0^\pi a(\theta)\, e^{-2\xi\theta}\ \sin^\lambda\theta\ d\theta \qquad \xi \in \mathbb{R},$$

where the function $\vartheta_\lambda(\xi)$ is given by (2.2).

We mention in this context (see, for example, [1, 3]) the Wick (or covariant, or Berezin) symbol $\widetilde{a}_\lambda(z,\overline{z})$, $z \in \Pi$, of the Toeplitz operator $T_a^{(\lambda)}$, which together with the so-called star product carries many essential properties of the corresponding Toeplitz operator. Recall that given a bounded operator A acting on a Hilbert space H which has a system of coherent states $\{k_g\}_{g\in G}$, its Wick symbol is defined as

$$\widetilde{a}_A(g,g) = \frac{\langle Ak_g, k_g\rangle}{\langle k_g, k_g\rangle}, \qquad g \in G.$$

In our particular case we have $A = T_a^{(\lambda)}$, $H = \mathcal{A}^2_\lambda(\Pi)$, and $k_g = k_z(\zeta) = i^{\lambda+2}(\zeta-\overline{z})^{-(\lambda+2)}$, where $z, \zeta \in \Pi$. The star product defines the composition of two Wick symbols $\widetilde{a}_A$ and $\widetilde{a}_B$ of the operators A and B, respectively, as the Wick symbol of the composition AB, i.e., $\widetilde{a}_A \star \widetilde{a}_B = \widetilde{a}_{AB}$.

In Section 3 we give the formulas for the Wick symbols of Toeplitz operators $T_a^{(\lambda)}$, whose symbols depend only on θ, and the formulas for the star product in terms of our function $\gamma_{a,\lambda}$.

An interesting and important feature of Toeplitz operators on the (weighted) Bergman spaces is that such operators can be *bounded* even when they have symbols *unbounded near the boundary.* In Section 4 we study in details boundedness properties of Toeplitz operators with such unbounded symbols. We give several separate sufficient and necessary boundedness conditions, as well as a number of illustrating examples. It turns out that for unbounded symbols, the behaviour of certain means of a symbol, rather than the behaviour of the symbol itself, plays a crucial role in the boundedness properties. Given a symbol a, it is natural to introduce the set $B(a)$ of values $\lambda \in [0,\infty)$ for which the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$. We show that being nonempty the set $B(a)$ may have only one of the following three forms: $[0,\infty)$, $[0,\nu)$, or $[0,\nu]$.

Section 5 is devoted to the spectral properties. The (continuous) spectrum of each $T_a^{(\lambda)}$ coincides with the closure of the image of the corresponding *continuous* function $\gamma_{a,\lambda}$. For each fixed λ the spectrum seems to be quite unrestricted, as the definite tendency starts appearing only as λ tends to infinity. The correspondence principle suggests that the limit set of the spectra has to be somehow connected with the range of the initial symbol a. This is definitely true for continuous symbols. Given a continuous symbol a, the limit set of the spectra, which we will denote by $M_\infty(a)$, does coincide with the range of a. As in [6], the new effects appear when we consider more complicated symbols. To understand the impact of each type of a discontinuity of a symbol we consider two model cases, piecewise continuous and oscillating symbols. In particular, in the case of piecewise continuous symbols the limit set $M_\infty(a)$ coincides with the range of a together with the *line segments connecting the one-sided limit points* of our piecewise continuous symbol.

Proofs of various theorems and construction of examples in the section are analogous to those of [5] and we omit them. On the other hand side to diminish somehow an imbalance with [5] we give a few illustrating graphical examples.

The authors would like to express their deep gratitude to O. Grudskaia for her help in preparation of the figures in the paper.

2. Representations of the weighted Bergman space

We start with the description of the weighted Bergman space $\mathcal{A}^2_\lambda(\Pi))$, where $\lambda \in (-1,+\infty)$, which is compatible with the polar coordinates in Π. Passing to polar coordinates we have

$$L_2(\Pi, d\mu_\lambda) = L_2(\mathbb{R}_+, r^{\lambda+1}dr) \otimes L_2([0,\pi], 1/\pi 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta),$$

Rewriting the equation $\frac{\partial}{\partial \bar{z}}\varphi = 0$ in polar coordinates, we have that the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ as the set of all functions satisfying the equation

$$\left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta}\right)\varphi(r,\theta) = 0.$$

Introduce the unitary operator

$$\begin{aligned} U_1 = 1/\sqrt{\pi}(M\otimes I) \quad &: \quad L_2(\Pi, d\mu_\lambda) = L_2(\mathbb{R}_+, r^{\lambda+1}dr) \otimes L_2([0,\pi], 1/\pi 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta) \\ &\longrightarrow \quad L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta), \end{aligned}$$

where the Mellin transform $M : L_2(\mathbb{R}_+, r^{\lambda+1}dr) \longrightarrow L_2(\mathbb{R})$ is given by

$$(M\psi)(\xi) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}_+} r^{-i\xi+\lambda/2}\ \psi(r)\,dr.$$

The inverse Mellin transform $M^{-1} : L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}_+, r^{\lambda+1}dr)$ has the form

$$(M^{-1}\psi)(r) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} r^{i\xi-\lambda/2-1}\, \psi(\xi)\; d\xi.$$

It is easy to see that

$$U_1 \left(r\frac{\partial}{\partial r} + i\frac{\partial}{\partial \theta} \right) U_1^{-1} = i(\xi + (\lambda/2+1)i)I + i\frac{\partial}{\partial \theta}.$$

Thus, the image of the Bergman space $\mathcal{A}^2_{1,\lambda} = U_1(\mathcal{A}^2_\lambda(\Pi))$ can be described as the (closed) subspace of $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ which consists of all functions $\varphi(\xi,\theta)$ satisfying the equation

$$\left((\xi + (\lambda/2+1)i)I + \frac{\partial}{\partial \theta} \right) \varphi(\xi,\theta) = 0.$$

The general $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ solution of this equation has the form

$$\varphi(\xi,\theta) = f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta}, \qquad f(\xi) \in L_2(\mathbb{R}), \tag{2.1}$$

where (see, for example, [4] formula 3.892)

$$\begin{aligned} \vartheta_\lambda(\xi) &= \left(2^\lambda(\lambda+1) \int_0^\pi e^{-2\xi\theta} \sin^\lambda\theta d\theta \right)^{-1/2} \\ &= \frac{\mathrm{B}(\frac{\lambda+2}{2}+i\xi, \frac{\lambda+2}{2}-i\xi)^{1/2}}{\sqrt{\pi}}\, e^{\pi\xi/2} = \frac{|\Gamma(\frac{\lambda+2}{2}+i\xi)|}{\sqrt{\pi}\,\Gamma(\lambda+2)^{1/2}}\, e^{\pi\xi/2}, \end{aligned} \tag{2.2}$$

and

$$\|\varphi(\xi,\theta)\|_{L_2(\mathbb{R})\otimes L_2([0,\pi],2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)} = \|f(\xi)\|_{L_2(\mathbb{R})}.$$

Lemma (2.3). *The unitary operator $U_1 = 1/\sqrt{\pi}(M \otimes I)$ is an isometric isomorphism of the space $L_2(\Pi, d\mu_\lambda)$, where $\lambda \in (-1,+\infty)$, onto $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ under which the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is mapped onto*

$$\mathcal{A}^2_{1,\lambda} = \left\{ \varphi(\xi,\theta) = f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta} : \quad f(\xi) \in L_2(\mathbb{R}) \right\}.$$

As above, let $R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2_{1,\lambda}(\Pi) \subset L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ be the isometric imbedding given by

$$(R_0 f)(\xi,\theta) = f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta}.$$

The adjoint operator $R_0^* : L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta) \longrightarrow L_2(\mathbb{R})$ has the form

$$(R_0^*\psi)(\xi) = 2^\lambda(\lambda+1)\vartheta_\lambda(\xi) \int_0^\pi \psi(\xi,\theta)\, e^{-(\xi-(1+\lambda/2)i)\theta}\, \sin^\lambda\theta\, d\theta,$$

and

$$\begin{aligned} R_0^* R_0 = I &\quad : \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R_0 R_0^* = B_1 &\quad : \quad L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta) \longrightarrow \mathcal{A}^2_{1,\lambda}, \end{aligned}$$

where $B_1 = U_1 B_\Pi^\lambda U_1^{-1}$ is the orthogonal projection of $L_2(\mathbb{R}) \otimes L_2([0,\pi], 2^\lambda(\lambda+1)\sin^\lambda\theta d\theta)$ onto $\mathcal{A}^2_{1,\lambda}$.

Now the operator $R_\lambda = R_0^* U_1$ maps the space $L_2(\Pi, d\mu_\lambda)$ onto $L_2(\mathbb{R})$, and its restriction

$$R_\lambda|_{\mathcal{A}^2_\lambda(\Pi)} : \mathcal{A}^2_\lambda(\Pi) \longrightarrow L_2(\mathbb{R})$$

is an isometric isomorphism. The adjoint operator

$$R_\lambda^* = U_1^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2_\lambda(\Pi) \subset L_2(\Pi, d\mu_\lambda)$$

is an isometric isomorphism of $L_2(\mathbb{R})$ onto $\mathcal{A}^2_\lambda(\Pi)$.

Remark (2.4). We have

$$\begin{aligned} R_\lambda R_\lambda^* = I \quad &: \quad L_2(\mathbb{R}) \longrightarrow L_2(\mathbb{R}), \\ R_\lambda^* R_\lambda = B_\Pi^\lambda \quad &: \quad L_{2,\lambda}(\Pi) \longrightarrow \mathcal{A}^2_\lambda(\Pi). \end{aligned}$$

Theorem (2.5). *The isometric isomorphism*

$$R_\lambda^* = U_1^* R_0 : L_2(\mathbb{R}) \longrightarrow \mathcal{A}^2_\lambda(\Pi)$$

is given by

$$(R_\lambda^* f)(z) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} z^{i\xi - (1+\lambda/2)}\, \vartheta_\lambda(\xi)\, f(\xi)\, d\xi. \tag{2.6}$$

Proof. Calculate

$$\begin{aligned} (R_\lambda^* f)(z) &= (U_1^* R_0 f)(z) \\ &= \sqrt{\pi}(M^{-1} \otimes I) f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi + (1+\lambda/2)i)\theta} \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} r^{i\xi - (1+\lambda/2)} f(\xi)\, \vartheta_\lambda(\xi)\, e^{-(\xi+(1+\lambda/2)i)\theta}\, d\xi \\ &= \frac{1}{\sqrt{2}} \int_{\mathbb{R}} z^{i\xi-(1+\lambda/2)}\, \vartheta_\lambda(\xi)\, f(\xi)\, d\xi. \end{aligned}$$

□

Corollary (2.7). *The inverse isomorphism*

$$R_\lambda : \mathcal{A}^2_\lambda(\Pi) \longrightarrow L_2(\mathbb{R})$$

is given by

$$(R_\lambda \varphi)(\xi) = \frac{\vartheta_\lambda(\xi)}{\sqrt{2}} \int_\Pi (\overline{z})^{-i\xi-(1+\lambda/2)}\, \varphi(z)\, \mu_\lambda(z)\, dv(z). \tag{2.8}$$

The above representation of the Bergman space $\mathcal{A}^2_\lambda(\Pi)$ is especially important in the study of the Toeplitz operators with symbols depending only on $\theta = \arg z$.

Theorem (2.9). *Given $a = a(\theta) \in L_1(0,\pi)$, the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I = R_\lambda\, T_a^{(\lambda)} R_\lambda^*$, acting on $L_2(\mathbb{R})$. The function $\gamma_{a,\lambda}(\xi)$ is given by*

$$\begin{aligned} \gamma_{a,\lambda}(\xi) &= 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \int_0^\pi a(\theta)\, e^{-2\xi\theta}\, \sin^\lambda\theta\, d\theta \\ &= \left(\int_0^\pi e^{-2\xi\theta} \sin^\lambda\theta\, d\theta\right)^{-1} \int_0^\pi a(\theta)\, e^{-2\xi\theta} \sin^\lambda\theta\, d\theta,\ \ \xi \in \mathbb{R}. \end{aligned} \tag{2.10}$$

Proof. Calculate

$$\begin{aligned} R_\lambda T_a^{(\lambda)} R_\lambda^* &= R_\lambda B_{\Pi,\lambda} a B_{\Pi,\lambda} R_\lambda^* = R_\lambda (R_\lambda^* R_\lambda) a (R_\lambda^* R_\lambda) R_\lambda^* \\ &= (R_\lambda R_\lambda^*) R_\lambda a R_\lambda^* (R_\lambda R_\lambda^*) = R_\lambda a R_\lambda^* \\ &= R_0^* U_1 a(\theta) U_1^{-1} R_0 \\ &= R_0^* a(\theta) R_0. \end{aligned}$$

Thus

$$\begin{aligned} (R_0^* a(\theta) R_0 f)(\xi) &= 2^\lambda (\lambda+1) \vartheta_\lambda(\xi) \int_0^\pi a(\theta)\, e^{-(\xi-(1+\lambda/2)i)\theta} f(\xi) \\ &\times \ \vartheta_\lambda(\xi)\ e^{-(\xi+(1+\lambda/2)i)\theta} \sin^\lambda \theta \, d\theta \\ &= \gamma_{a,\lambda}(\xi) f(\xi), \end{aligned}$$

where

$$\gamma_{a,\lambda}(\xi) = 2^\lambda \ (\lambda+1) \vartheta_\lambda^2(\xi) \int_0^\pi a(\theta)\, e^{-2\xi\theta} \sin^\lambda \theta \ d\theta \qquad \xi \in \mathbb{R}.$$

Here the function $\vartheta_\lambda(\xi)$ is given by (2.2). □

The above theorem suggests considering not only L_∞-symbols, but unbounded ones as well. Note that given a bounded symbol $a(z)$, the Toeplitz operator $T_a^{(\lambda)}$ is bounded on all spaces $\mathcal{A}^2_\lambda(\Pi)$, for $\lambda \in (-1,\infty)$, and the corresponding norms are uniformly bounded by $\sup_z |a(z)|$. That is, all spaces $\mathcal{A}^2_\lambda(\Pi)$, where $\lambda \in (-1,\infty)$, are natural and appropriate for Toeplitz operators with *bounded* symbols. As one of our aims is a systematic study of *unbounded* symbols, we wish to have a sufficiently large class of them common to all admissible λ; moreover, we are especially interested in properties of Toeplitz operators for large values of λ. Thus it is convenient for us to consider λ belonging only to $[0,\infty)$, which we will always assume in what follows.

We have obviously:

Corollary (2.11). *The Toeplitz operator $T_a^{(\lambda)}$ with symbol $a(\theta)$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$ if and only if the corresponding function $\gamma_{a,\lambda}(\xi)$ is bounded.*

3. Toeplitz operators with symbols depending on $\theta = \arg z$

Reverting the statement of Theorem 2.9 we come to the following spectral-type representation of a Toeplitz operator.

Theorem (3.1). *Let $a = a(\theta) \in L_1(0,\pi)$. Then the Toeplitz operator $T_a^{(\lambda)}$ acting on $\mathcal{A}^2_\lambda(\Pi)$ admits the representation*

$$(T_a^{(\lambda)}\varphi)(z) = \frac{1}{\sqrt{2}} \int_{\mathbb{R}} z^{i\xi-(1+\lambda/2)}\, \vartheta_\lambda(\xi)\, \gamma_{a,\lambda}(\xi)\, f(\xi)\, d\xi, \tag{3.2}$$

where $f(\xi) = (R_\lambda\varphi)(\xi) \in L_2(\mathbb{R})$.

Proof. Follows directly from Theorems 2.9, and 2.5, and Corollary 2.7. □

Theorem (3.3). *Given $a = a(\theta) \in L_1(0,\pi)$, the Wick symbol $\widetilde{a}_\lambda(z,\overline{z})$ of the Toeplitz operator $T_a^{(\lambda)}$ depends only on $\theta\,(=\arg z\,)$ and has the form*

$$\widetilde{a}_\lambda(\theta) = \widetilde{a}_\lambda(z,\overline{z}) = 2^{\lambda+1} \sin^{\lambda+2}\theta \int_{\mathbb{R}} e^{-2\xi\theta}\, \vartheta_\lambda^2(\xi)\, \gamma_{a,\lambda}(\xi)\, d\xi \tag{3.4}$$

and the corresponding Wick function is given by formula

$$\begin{aligned}
\widetilde{a}_\lambda(z,\overline{w}) &= \frac{\langle T_a^{(\lambda)} k_w, k_z\rangle}{\langle k_w, k_z\rangle} \\
&= (z-\overline{w})^{\lambda+2}\,(z\overline{w})^{-(\lambda+2)/2}\,\frac{i^{-(\lambda+2)}}{2}\int_{\mathbb{R}}(\frac{z}{\overline{w}})^{i\xi}\,\vartheta_\lambda^2(\xi)\,\gamma_{a,\lambda}(\xi)\,d\xi. \qquad (3.5)
\end{aligned}$$

Proof. Consider $k_z(w) = i^{2+\lambda}(w-\overline{z})^{-(\lambda+2)} = i^{2+\lambda}(\rho e^{i\alpha} - re^{-i\theta})^{-(\lambda+2)}$ and calculate

$$(U_1 k_z)(\xi,\alpha) = \frac{i^{2+\lambda}}{\pi\sqrt{2}}\int_{\mathbb{R}_+}\rho^{-i\xi+\lambda/2}(\rho e^{i\alpha}-\overline{z})^{-(\lambda+2)}d\rho.$$

Using formula 3.194.3 from [4] and (2.2), we have

$$\begin{aligned}
(U_1k_z)(\xi,\alpha) &= \frac{\mathrm{B}(\frac{\lambda+2}{2}-i\xi,\frac{\lambda+2}{2}+i\xi)}{\sqrt{2}\pi}\,e^{\pi\xi}\,e^{-\xi\alpha-i\frac{\lambda+2}{2}\alpha}\,(\overline{z})^{-i\xi-\frac{\lambda+2}{2}} \\
&= \frac{\vartheta_\lambda^2(\xi)}{\sqrt{2}}\,e^{-\xi\alpha-i\frac{\lambda+2}{2}\alpha}\,(\overline{z})^{-i\xi-\frac{\lambda+2}{2}}.
\end{aligned}$$

Thus

$$\begin{aligned}
\langle T_a^{(\lambda)}k_z,k_z\rangle &= \langle ak_z,k_z\rangle = \langle U_1ak_z,U_1k_z\rangle = \langle aU_1k_z,U_1k_z\rangle = \\
&= \frac{1}{2}\int_{\mathbb{R}}\int_0^\pi a(\alpha)\,\vartheta_\lambda^4(\xi)e^{-2\xi\alpha}(\overline{z})^{-i\xi-\frac{\lambda+2}{2}}\,z^{i\xi-\frac{\lambda+2}{2}}2^\lambda(\lambda+1)\sin^\lambda\alpha\,d\xi d\alpha \\
&= \frac{r^{-(\lambda+2)}}{2}\int_{\mathbb{R}}\vartheta_\lambda^2(\xi)\,e^{-2\xi\theta}d\xi\;2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\int_0^\pi a(\alpha)\,e^{-2\xi\alpha}\,\sin^\lambda\alpha\,d\alpha \\
&= \frac{r^{-(\lambda+2)}}{2}\int_{\mathbb{R}}\vartheta_\lambda^2(\xi)\,e^{-2\xi\theta}\,\gamma_{a,\lambda}(\xi)d\xi.
\end{aligned}$$

Similarly

$$\langle T_a^{(\lambda)}k_w,k_z\rangle = \frac{(z\overline{w})^{-(\lambda+2)/2}}{2}\int_{\mathbb{R}}(\frac{z}{\overline{w}})^{i\xi}\,\vartheta_\lambda^2(\xi)\,\gamma_{a,\lambda}(\xi)\,d\xi.$$

Furthermore $\langle k_w,k_z\rangle = k_w(z) = i^{\lambda+2}(z-\overline{w})^{-(\lambda+2)}$, and $\langle k_z,k_z\rangle = k_z(z) = (2\mathrm{Im}\,z)^{-(\lambda+2)}$. Thus we have both (3.4) and (3.5). □

Remark (3.6). Given a symbol $a = a(\theta)\in L_1(0,\pi)$, writing the Toeplitz operator $T_a^{(\lambda)}$ in terms of its Wick symbol we obtain formula (3.2). Indeed

$$\begin{aligned}
(T_a^{(\lambda)}\varphi)(z) &= \int_\Pi \widetilde{a}(z,\overline{w})\,\frac{\varphi(w)\,i^{\lambda+2}}{(z-\overline{w})^{\lambda+2}}\,\mu_\lambda(w)\,dv(w) \\
&= \frac{1}{2}\int_\Pi(z\overline{w})^{-(\lambda+2)/2}\varphi(w)\,\mu_\lambda(w)\,dv(w)\int_{\mathbb{R}}(\frac{z}{\overline{w}})^{i\xi}\,\vartheta_\lambda^2(\xi)\,\gamma_{a,\lambda}(\xi)\,d\xi \\
&= \frac{1}{\sqrt{2}}\int_{\mathbb{R}}z^{i\xi-\frac{\lambda+2}{2}}\,\vartheta_\lambda(\xi)\,\gamma_{a,\lambda}(\xi)\,d\xi \\
&\times\;\frac{\vartheta_\lambda(\xi)}{\sqrt{2}}\int_\Pi(\overline{w})^{-i\xi-\frac{\lambda+2}{2}}\,\varphi(w)\,\mu_\lambda(w)\,dv(w) \\
&= \frac{1}{\sqrt{2}}\int_{\mathbb{R}}z^{i\xi-\frac{\lambda+2}{2}}\,\vartheta_\lambda(\xi)\,\gamma_{a,\lambda}(\xi)\,(R_\lambda\varphi)(\xi)\,d\xi.
\end{aligned}$$

Corollary (3.7). *Let $T_a^{(\lambda)}$ and $T_b^{(\lambda)}$ be two Toeplitz operators with symbols $a(\theta), b(\theta) \in L_1(0,\pi)$ respectively, and let $\widetilde{a}_\lambda(\theta)$ and $\widetilde{b}_\lambda(\theta)$ be their Wick symbols. Then the Wick symbol $\widetilde{c}(\theta)$ of the composition $T_a^{(\lambda)}T_b^{(\lambda)}$ is given by*

$$\widetilde{c}_\lambda(\theta) = (\widetilde{a}_\lambda \star \widetilde{b}_\lambda)(\theta) = 2^{\lambda+1}\sin^{\lambda+2}\theta \int_{\mathbb{R}} e^{-2\xi\theta}\,\vartheta_\lambda^2(\xi)\,\gamma_{a,\lambda}(\xi)\,\gamma_{b,\lambda}(\xi)\,d\xi.$$

Proof. This can be verified directly from the formula for the star product, and also follows immediately from Theorems 2.9 and 3.3. □

4. Boundedness of Toeplitz operators with symbols depending on $\theta = \arg z$.

Recall (Corollary 2.11) that the function

$$\gamma_{a,\lambda}(\xi) = \left(\int_0^\pi e^{-2\xi\theta}\ \sin^\lambda\theta\,d\theta\right)^{-1}\int_0^\pi a(\theta)\ e^{-2\xi\theta}\ \sin^\lambda\theta\,d\theta, \qquad \xi\in\mathbb{R} \tag{4.1}$$

is responsable for the boundedness of a Toeplitz operator with symbol $a(\theta)$ ($\in L_1(0,\pi)$). If the symbol $a(\theta) \in L_\infty(0,\pi)$, then the operator $T_a^{(\lambda)}$ is obviously bounded on $\mathcal{A}_\lambda^2(\Pi)$ for each λ, and $\|T_a^{(\lambda)}\| \le \text{ess-sup}|a(\theta)|$.

For $a(\theta) \in L_1(0,\pi)$ the function $\gamma_{a,\lambda}(\xi)$ is continuous at all finite points $\xi \in \mathbb{R}$. For a "very large ξ" ($\xi \to +\infty$) the exponent $e^{-2\xi\theta}$ has a very sharp maximum at the point $\theta = 0$, and thus the major contribution to the integral containing $a(\theta)$ in (4.1) for these "very large ξ" is determined by values of $a(\theta)$ int a neighborhood of the point 0. The major contribution for a "very large negative ξ" ($\xi \to -\infty$) is determined by values of $a(\theta)$ at a neighborhood of π, due to a very sharp maximum of $e^{-2\xi\theta}$ at $\theta = \pi$ for these values of ξ. In particular, if $a(\theta)$ has limits at the points 0 and π, then

$$\begin{aligned}\lim_{\xi\to+\infty}\gamma_{a,\lambda}(\xi) &= \lim_{\theta\to 0} a(\theta),\\ \lim_{\xi\to-\infty}\gamma_{a,\lambda}(\xi) &= \lim_{\theta\to\pi} a(\theta).\end{aligned}$$

As a matter of fact, 0 and π are the only worrying points for unbounded symbols $a(\theta) \in L_1(0,\pi)$. Moreover, the behaviour of certain means of a symbol, rather than the behaviour of the symbol itself, plays a crucial role under the study of boundedness properties.

Given $\lambda \in [0,\infty)$ and a function $a(\theta) \in L_1(0,\pi)$ introduce the following means:

$$\begin{aligned}C_{a,\lambda}^{(1)}(\sigma) &= \int_0^\sigma a(\theta)\sin^\lambda\theta d\theta,\\ D_{a,\lambda}^{(1)}(\sigma) &= \int_\sigma^\pi a(\theta)\sin^\lambda\theta d\theta,\\ C_{a,\lambda}^{(j)}(\sigma) &= \int_0^\sigma C_{a,\lambda}^{(j-1)}(\theta)d\theta, \quad j = 2,3,\ldots,\\ D_{a,\lambda}^{(j)}(\sigma) &= \int_\sigma^\pi D_{a,\lambda}^{(j-1)}(\theta)d\theta, \quad j = 2,3,\ldots.\end{aligned}$$

Theorem (4.2). *Let $a(\theta) \in L_1(0,\pi)$. If for certain $\lambda_0 \in [0,\infty)$ and $j_0, j_1 \in \mathbb{N}$ the following conditions hold*

$$C^{(j_0)}_{a,\lambda_0}(\sigma) = O(\sigma^{j_0+\lambda_0}), \qquad \sigma \to 0, \tag{4.3}$$

$$D^{(j_1)}_{a,\lambda_0}(\sigma) = O((\pi-\sigma)^{j_1+\lambda_0}), \qquad \sigma \to \pi, \tag{4.4}$$

then the corresponding Toeplitz operator $T^{(\lambda)}_a$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$ for each $\lambda \in [\lambda_0, \infty)$.

Proof. Note that the function $\gamma_{a,\lambda}(\xi)$ is continuous at finite points. Let $\xi \to +\infty$ and the condition (4.3) holds with $j_0 = 1$. Then

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= 2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)\int_0^\pi \sin^{\lambda-\lambda_0}(\theta)e^{-2\xi\theta}dC^{(1)}_{a,\lambda_0}(\theta)\\
&= 2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)\left|\int_0^\pi C^{(1)}_{a,\lambda_0}(\theta)[(\lambda-\lambda_0)\sin^{\lambda-\lambda_0-1}\theta\cos\theta\right.\\
&- \left. 2\xi\sin^{\lambda-\lambda_0}\theta]e^{-2\xi\theta}d\theta\right|\\
&\le \operatorname{const} 2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)\left[(\lambda-\lambda_0)\int_0^\infty \theta^\lambda e^{-2\xi\theta}d\theta + 2\xi\int_0^\infty \theta^{\lambda+1}e^{-2\xi\theta}d\theta\right]\\
&\le \operatorname{const}\vartheta^2_\lambda(\xi)\left[(\lambda-\lambda_0)(2\xi)^{-(\lambda+1)}\Gamma(\lambda+1) + (2\xi)^{-(\lambda+1)}\Gamma(\lambda+2)\right]\\
&\le \operatorname{const}(2\lambda-\lambda_0+1)2^\lambda(\lambda+1)\vartheta^2_\lambda(\xi)(2\xi)^{-(\lambda+1)}\Gamma(\lambda+1).
\end{aligned}$$

It is easy to get the asymptotic representation of the function $\vartheta^2_\lambda(\xi)$. According to (2.2) we have

$$\begin{aligned}
2^{-\lambda}(\lambda+1)^{-1}\vartheta^{-2}_\lambda(\xi) &= \int_0^\pi e^{-2\xi\theta}\sin^\lambda\theta d\theta\\
&= \int_0^\pi \theta^\lambda e^{-2\xi\theta}d\theta[1+\theta(\xi^{-1})]\\
&= (2\xi)^{-(\lambda+1)}\Gamma(\lambda+1)[1+O(\xi^{-1})]. \qquad (4.5)
\end{aligned}$$

Thus we have finally

$$|\gamma_{a,\lambda}(\xi)| \le \operatorname{const}(2\lambda-\lambda_0+1).$$

The case $\xi \to -\infty$ (and $j_1 = 1$) is reduced to the previous one using the change of variable $\theta = \pi - \theta'$ in the integral for $\gamma_{a,\lambda}(\xi)$.

The cases $j_{0,1} > 1$ are considered analogously using integration by parts. □

The proof of the following statement is analogous to that of Theorem 4.3 in [5].

Theorem (4.6). 1. *Let conditions (4.3), (4.4) hold for $j_0 = j_0'$, $j_1 = j_1'$, and some λ_0. Then these conditions hold for $j_0 = j_0'+1$, $j_1 = j_1'+1$, and the same λ_0.*
2. *Let conditions (4.3), (4.4) hold for $j_0 = j_0'$, $j_1 = j_1'$, and some λ_0. Then these conditions hold for $j_0 = j_0'$, $j_1 = j_1'$, and λ_0 replaced by any $\lambda_1 \ge \lambda_0$.*

EXAMPLE (4.7). *Consider the following family of unbounded symbols*

$$a(\theta) = (\sin\theta)^{-\beta} \sin[(\sin\theta)^{-\alpha}].$$

As in Example 4.4 in [5] *it can be proved that for all* $\lambda \geq 0$ *the operator* $T_a^{(\lambda)}$ *is bounded for each* $\beta \in (0,1)$ *and* $\alpha > 0$.

THEOREM (4.8). *Let the Toeplitz operator* $T_a^{(\lambda)}$, *with* $a(\theta) \in L_1(0,\pi)$, *be bounded on some* $\mathcal{A}^2_{\lambda_0}(\Pi)$. *Then it is bounded on each* $\mathcal{A}^2_\lambda(\Pi)$, *with* $\lambda \in [0, \lambda_0]$.

Proof. Let $\sup_{\xi\in\mathbb{R}} |\gamma_{a,\lambda_0}(\xi)| < \infty$. We split $a(\theta)$ in two functions which vanish on neighborhoods of 0 and π, respectively. The study of these two cases is quite similar, thus we suppose that $a(\theta)$ vanishes in a neighborhood of π, for example. Suppose also that $\xi \to \infty$. A similar argument is applicable for the study of the behavior of $\gamma_{a,\lambda}(\xi)$ when $\xi \to -\infty$. For $\lambda \in [0, \lambda_0)$, write

$$\gamma_{a,\lambda}(\xi) = \frac{2^{2\lambda-\lambda_0}\ (\lambda+1)\vartheta^2_\lambda(\xi)}{\Gamma(\lambda_0-\lambda)} \int_0^\infty y^{\lambda_0-\lambda-1} dy \int_0^\pi a(\theta) e^{-2\theta(\xi+\frac{\sin\theta}{\theta}y)}\ \sin^{\lambda_0}\theta d\theta.$$

Using $\frac{\sin\theta}{\theta} = 1 + O(\theta^2)$, as $\theta \to 0$, for some $c_\lambda \neq 0$, we have

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= (c_\lambda + o(1))\vartheta^2_\lambda(\xi) \int_0^\infty y^{\lambda_0-\lambda-1} dy \int_0^\pi a(\theta) e^{-2\theta(\xi+y)}\ \sin^{\lambda_0}\theta d\theta \\
&= \frac{(c_\lambda + o(1))\vartheta^2_\lambda(\xi)}{2^{\lambda_0}(\lambda_0+1)} \int_0^\infty y^{\lambda_0-\lambda-1} \frac{\gamma_{a,\lambda_0}(\xi+y)}{\vartheta^2_{\lambda_0}(\xi+y)} dy.
\end{aligned}$$

Using (4.5) and $\sup_{\xi\in\mathbb{R}} |\gamma_{a,\lambda_0}(\xi)| < \infty$ we have

$$\begin{aligned}
|\gamma_{a,\lambda}(\xi)| &\leq \text{const}\ \xi^{\lambda+1} \int_0^\infty y^{\lambda_0-\lambda-1} (\xi+y)^{-(\lambda_0+1)} dy \\
&= \text{const} \int_0^\infty u^{\lambda_0-\lambda-1} (1+u)^{-(\lambda_0+1)} du < \infty,
\end{aligned}$$

since $\lambda < \lambda_0$ and $\lambda_0 + 1 > 1$. □

As an immediate corollary of Theorems 4.2 and 4.8 we have now.

THEOREM (4.9). *Under the hypothesis of Theorem 4.2 the Toeplitz operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$ *for each* $\lambda \in [0, \infty)$.

The proof of the next theorem is analogous to one of Theorem 4.8 in [5].

THEOREM (4.10).
1. *Assume that* $a(\theta) \in L_1(0,\pi)$ *and* $a(\theta) \geq 0$ *almost everywhere. Let the operator* $T_a^{(\lambda')}$ *be bounded on* $\mathcal{A}^2_{\lambda'}(\Pi)$ *for some* $\lambda' > 0$. *Then the conditions (4.3) and (4.4) hold for* $j_0 = j_1 = 1$, $\lambda_0 = 0$ *and consequently the operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$ *for arbitrary* $\lambda \in [0, \infty)$.
2. *Assume that the means satisfy* $C^{(j_0)}_{a,\mu_0}(\sigma) \geq 0$ *and* $D^{(j_1)}_{a,\mu_1}(\sigma) \geq 0$ *almost everywhere for some* $j_0 \geq 1$, $j_1 \geq 1$ *and* $\mu_0 \geq 0$, $\mu_1 \geq 0$, *and that the operator* $T_a^{(\lambda')}$ *is bounded on* $\mathcal{A}^2_{\lambda'}(\Pi)$ *for some* $\lambda' \geq 0$. *Then the operator* $T_a^{(\lambda)}$ *is bounded on* $\mathcal{A}^2_\lambda(\Pi)$ *for arbitrary* $\lambda \in [0, \infty)$.

For a nonnegative $a(\theta)$ we set

$$\begin{aligned} m_{a,0}(\sigma) &= \text{ess-inf}_{\theta\in(0,\sigma)} a(\theta), \\ m_{a,\pi}(\sigma) &= \text{ess-inf}_{\theta\in(\sigma,\pi)} a(\theta). \end{aligned}$$

COROLLARY (4.11). *Given a nonnegative symbol, if either* $\lim_{\sigma\to 0} m_{a,0}(\sigma) = \infty$ *or* $\lim_{\sigma\to\pi} m_{a,\pi}(\sigma) = \infty$*, then the Toeplitz operator* $T_a^{(\lambda)}$ *is unbounded on each* $\mathcal{A}^2_\lambda(\Pi)$*, with* $\lambda \in [0,\infty)$.

For a symbol $a(\theta) \in L_1(0,\pi)$ we denote by $\widetilde{B}(a)$ the set of points $\lambda \in [0,\infty)$ for which the corresponding Toeplitz operator $T_a^{(\lambda)}$ is bounded on $\mathcal{A}^2_\lambda(\Pi)$. Like in the parabolic case we have the following result, the proof of which is analogous to one of [5].

THEOREM (4.12). *There exists a family of symbols* $a_{\nu,\beta}(\theta)$*, where* $\nu \in (0,1)$*,* $\beta \in \mathbb{R}$*, such that*

a) $\widetilde{B}(a_{\nu,0}) = [0,\nu], \quad \beta = 0;$
b) $\widetilde{B}(a_{\nu,\beta}) = [0,\nu), \quad \beta > 0.$

5. Spectra of Toeplitz operators with symbols depending on $\theta = \arg z$

(5.1) Continuous symbols. Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point (typically $E = (0,+\infty)$), and suppose that, for each $\lambda \in E$, we are given a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,
(ii) $\lim_{n\to\infty} \lambda_n = +\infty$,
(iii) $z = \lim_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of the family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

For the case when E is a discrete set with a unique limit point at infinity, the above notion coincides with the partial limit set introduced in [7], Section 3.1.1. Following the arguments of Proposition 3.5 in [7] one can show that

$$M_\infty = \bigcap_\lambda \operatorname{clos}\left(\bigcup_{\mu\geq\lambda} M_\mu\right).$$

Note that obviously

$$\lim_{\lambda\to+\infty} M_\lambda = \lim_{\lambda\to+\infty} \overline{M}_\lambda = M_\infty.$$

The *a priori* spectral information for L_∞-symbols (see, for example, [1], [2]) says that for each $a \in L_\infty(\Pi)$ and each $\lambda \geq 0$

$$\operatorname{sp} T_a^{(\lambda)} \subset \operatorname{conv}(\text{ess-Range}\, a). \tag{5.1}$$

Given a symbol $a = a(\theta)$, the Toeplitz operator $T_a^{(\lambda)}$ acting on the space $\mathcal{A}_\lambda^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, where the function $\gamma_{a,\lambda}(\xi)$, $\xi \in \mathbb{R}$, is given by (2.10). Thus we have obviously

$$\operatorname{sp} T_a^{(\lambda)} = \overline{M_\lambda(a)},$$

where $M_\lambda(a) = \operatorname{Range} \gamma_{a,\lambda}$.

THEOREM (5.2). *Let $a = a(\theta) \in C[0,\pi]$. Then*

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{Range} a. \tag{5.3}$$

Proof. We find the asymptotic of the function $\gamma_{a,\lambda}(\xi)$ when $\lambda \to \pm\infty$ using the Laplace method. Introduce the large parameter $L = \sqrt{\lambda^2 + (2\xi)^2}$ and represent $\gamma_{a,\lambda}(\xi)$ in the form

$$\gamma_{a,\lambda}(\xi) = 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\int_0^\pi a(\theta)e^{-LS(\theta,\varphi)}d\theta,$$

where

$$S(\theta,\varphi) = \sin\varphi \ln(\sin\theta)^{-1} + (\cos\varphi)\theta,$$
$$\sin\varphi = \lambda/L, \quad \cos\varphi = 2\xi/L \quad \text{with } \varphi \in [0,\pi).$$

To find the point of minimum of $S(\theta,\varphi)$ calculate

$$S'_\theta(\theta,\varphi) = -(\sin\varphi)\cot\theta + \cos\varphi.$$

It is obvious that $S'_\theta(\theta_\varphi,\varphi) = 0$, for $\theta_\varphi \in (0,\pi)$, if and only if $\theta_\varphi = \varphi$.

Rewrite (5.3) in the form

$$\begin{aligned}\gamma_{a,\lambda}(\xi) - a(\varphi) &= 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\Bigg[\int_{U(\varphi)\cap[0,\pi]}(a(\theta)-a(\varphi))e^{-LS(\theta,\varphi)}d\theta \\ &+ \int_{[0,\pi]\setminus U(\varphi)}(a(\theta)-a(\varphi))e^{-LS(\theta,\varphi)}d\theta\Bigg] \equiv I_1(L) + I_2(L)\end{aligned}$$

where $U(\varphi)$ is a neighborhood of φ such that $\sup_{\theta\in U(\varphi)}|a(\theta)-a(\varphi)| < \varepsilon$ for sufficiently small ε. We have used

$$2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\int_0^\pi a(\varphi)e^{-LS(\theta,\varphi)}d\theta = a(\varphi).$$

Further,

$$\left|\int_{U(\varphi)}(a(\theta)-a(\varphi))e^{-LS(\theta,\varphi)}d\theta\right| \le \varepsilon\int_{U(\varphi)} e^{-LS(\theta,\varphi)}d\theta \le \varepsilon\left(2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\right)^{-1}$$

and finally,

$$\begin{aligned}\left|\int_{[0,\pi]\setminus U(\varphi)}(a(\theta)-a(\varphi))e^{-LS(\theta,\varphi)}d\theta\right| &\le 2\sup_{\theta\in[0,\pi]}|a(\theta)|\int_{[0,\pi]\setminus U(\varphi)} e^{-LS(\theta,\varphi)}d\theta \\ &\le (2M\sup_{\theta\in[0,\pi]}|a(\theta)|e^{-L\sigma(\varepsilon)})\left(2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)\right)^{-1},\end{aligned}$$

where $\sigma(\varepsilon) = \min_{\theta\in[0,\pi]\setminus U(\varphi)}(S(\theta,\varphi) - S(\varphi,\varphi))$. We note that $\sigma(\varepsilon)$ and M can be taken independent on $\varphi \in (0,\pi)$.

Since ε can be arbitrary small uniformly for $\varphi \in (0,\pi)$, we have

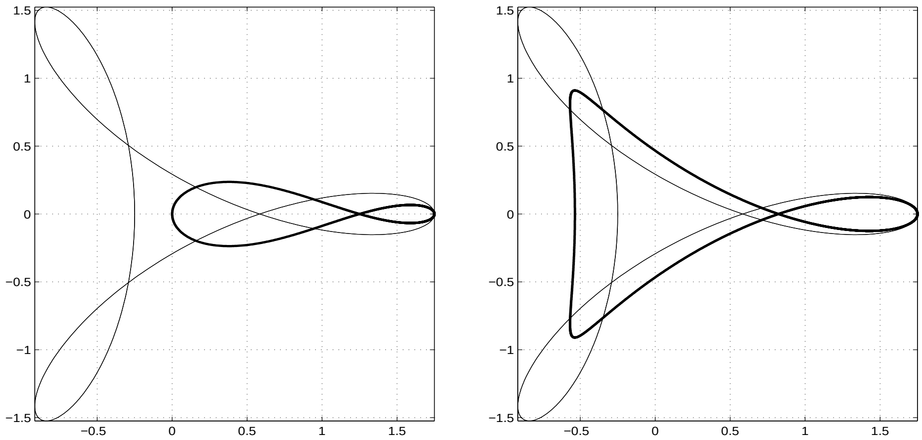
$$\gamma_{a,\lambda}(u) = a(\varphi)(1+\alpha(L)) \tag{5.4}$$

where $\lim_{L\to\infty} \alpha(L) = 0$ uniformly for $\varphi \in (0, \pi)$, which proves the theorem. □

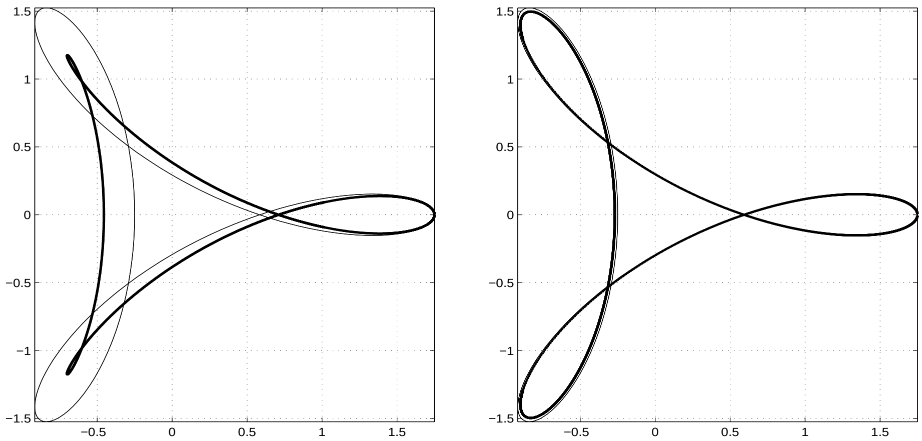
We illustrate the theorem on the continuous symbol (hypocycloid)

$$a(\theta) = \frac{3}{4} e^{4i\theta} + e^{-2i\theta},$$

and show the image of $\gamma_{a,\lambda}$ for the following values of λ: 0, 5, 12, and 200.



The function $\gamma_{a,\lambda}$ for $\lambda = 0$ and $\lambda = 5$.



The function $\gamma_{a,\lambda}$ for $\lambda = 12$ and $\lambda = 200$.

(5.2) Piecewise continuous symbols. Let $a(\theta)$ be a piecewise continuous function having jumps on a finite set of points $\{\theta_j\}_{j=1}^m$ where

$$\theta_0 = 0 < \theta_1 < \theta_2 < \ldots < \theta_m < \pi = \theta_{m+1},$$

and $a(\theta_j \pm 0), j = 1, \ldots, m,$ exist. Introduce the sets

$$J_j(a) := \{z \in \mathbb{C} : z = a(\theta),\ \theta \in (\theta_j, \theta_{j+1})\}$$

where $j = 0, \dots, m$, and let $I_j(a)$ be the segment with the endpoints $a(\theta_j - 0)$ and $a(\theta_j + 0)$, $j = 1, 2, \dots m$. We set

$$\widetilde{R}(a) = \left(\bigcup_{j=0}^{m} J_j(a) \right) \cup \left(\bigcup_{j=1}^{m} I_j(a) \right).$$

THEOREM (5.5). *Let $a(\theta)$ be a piecewise continuous function. Then*

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

Proof. We use the Laplace method as in Theorem 5.2. For any $\varepsilon > 0$ we take $\delta > 0$ such that for each interval $I \subset (\theta_j, \theta_{j+1})$ with length less then δ, $j = 1, 2, \dots, m$, the following inequality holds

$$\sup_{s_1, s_2 \in I} |a(s_1) - a(s_2)| < \varepsilon.$$

Suppose first that the minimum point $s_\varphi = \varphi$ satisfies the condition

$$\inf_{j=1,2,\dots,m} |\varphi - \theta_j| > \delta.$$

We have

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= a(\varphi) + 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \int_{\varphi-\delta}^{\varphi+\delta} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \\
&+ 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \int_{[0,\pi]\setminus(\varphi-\delta,\varphi+\delta)} (a(\theta) - a(\varphi)) e^{-LS(\theta,\varphi)} d\theta \\
&= a(\varphi) + O(\varepsilon) + O(e^{-\sigma L})
\end{aligned} \tag{5.6}$$

where

$$\sigma = \min_{[0,\pi]\setminus(\varphi-\delta,\varphi+\delta)} (S(\theta,\varphi) - S(\varphi,\varphi)).$$

Thus varying $\varphi \in \cup_{j=0}^{m} (\theta_j, \theta_{j+1})$ we have that

$$J_j(a) \subset M_\infty(a), \quad j = 0, 1, \dots, m.$$

Now suppose that there exist j such that $|\varphi - \theta_j| < \delta$. Then we have

$$\begin{aligned}
\gamma_{a,\lambda}(\xi) &= 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \left(a(\theta_j - 0) \int_{\varphi-\delta}^{\theta_j} e^{-LS(\theta,\varphi)} d\theta \right. \\
&+ \left. a(\theta_j + 0) \int_{\theta_j}^{\varphi+\delta} e^{-LS(\theta,\varphi)} d\theta \right) \\
&+ 2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi) \left(\int_{\varphi-\delta}^{\theta_j} (a(\theta) - a(\theta_j - 0)) e^{-LS(\theta,\varphi)} d\theta \right. \\
&+ \int_{\theta_j}^{\varphi+\delta} (a(\theta) - a(\theta_j + 0)) e^{-LS(\theta,\varphi)} d\theta \\
&+ \left. \int_{(0,\pi)\setminus(\varphi-\delta,\varphi+\delta)} a(\theta) e^{-LS(\theta,\varphi)} d\theta \right).
\end{aligned}$$

Taking δ small enough we suppose that

$$\frac{\theta_1}{2} < s_\varphi (= \varphi) < \frac{\pi + \theta_m}{2}.$$

Thus the function

$$(S''_{\theta,\theta}(\varphi, \varphi))^{-1} = -\sin\varphi$$

is uniformly bounded on φ and the following asymptotic calculations are uniform on φ:

$$\begin{aligned}
[2^\lambda(\lambda+1)\vartheta_\lambda^2(\xi)]^{-1} &= \int_0^\pi e^{-LS(\theta,\varphi)} d\theta \\
&= e^{-LS(\varphi,\varphi)} \int_0^\pi e^{-\frac{L}{2}(\sin^{-1}\varphi)(\theta-\varphi)^2} d\theta(1+O(1)) \\
&= e^{-LS(\varphi,\varphi)} \int_{-\varphi}^{\pi-\varphi} e^{-\frac{L}{2}(\sin^{-1}\varphi)u^2} du(1+O(1)) \\
&= \sqrt{2\sin\varphi}\, \frac{e^{-LS(\varphi,\varphi)}}{L^{1/2}} \int_{-\infty}^{\infty} e^{-v^2} dv(1+O(1)). \qquad (5.7)
\end{aligned}$$

Analogously

$$\int_{\theta_j}^{\varphi+\delta} e^{-LS(\theta,\varphi)} d\theta = \sqrt{2\sin\varphi}\, \frac{e^{-LS(\varphi,\varphi)}}{L^{1/2}} \int_{x_j}^{\infty} e^{-v^2} dv(1+O(1)) \qquad (5.8)$$

and

$$\int_{\varphi-\delta}^{\theta_j} e^{-LS(\theta,\varphi)} d\theta = \sqrt{2\sin\theta}\, \frac{e^{-LS(\varphi,\varphi)}}{L^{1/2}} \int_{-\infty}^{x_j} e^{-v^2} dv(1+O(1)), \qquad (5.9)$$

where

$$x_j = \left(\frac{L}{2\sin\varphi}\right)^{1/2} (\theta_j - \varphi).$$

Thus from (5.7)–(5.9) we have

$$\gamma_{a,\lambda}(\xi) = (a(\theta_j - 0)t + a(\theta_j + 0)\tau)(1 + O(1) + O(\varepsilon) + O(e^{-i\sigma})), \qquad (5.10)$$

where

$$t = (\int_{-\infty}^{x_j} e^{-v^2} dv)/(\int_{-\infty}^{\infty} e^{-v^2} dv) \quad \text{and} \quad \tau = (\int_{x_j}^{\infty} e^{-v^2} dv)/(\int_{-\infty}^{\infty} e^{-v^2} dv).$$

Now it is evident that $t, \tau \in [0,1]$ and $\tau + t = 1$, which implies $I_j(a) \subset M_\infty(a)$. Thus

$$\widetilde{R}(a) \subset M_\infty(a).$$

Representations (5.6) and (5.10) imply the inverse inclusion

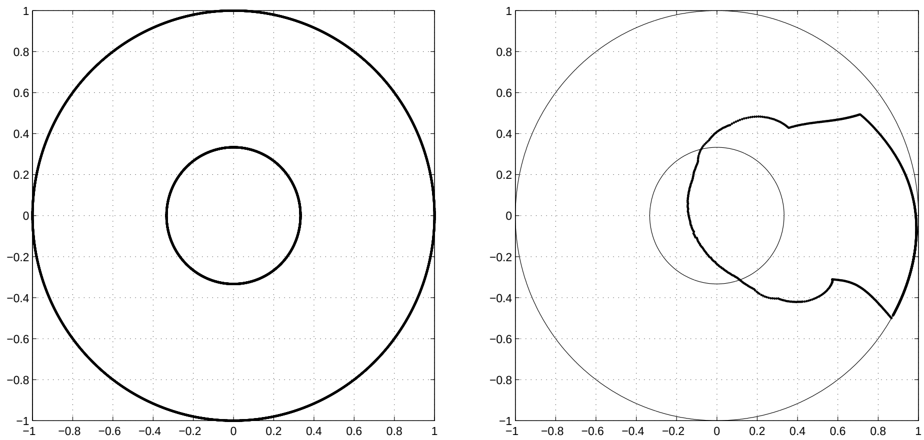
$$\widetilde{R}(a) \supset M_\infty(a).$$

$\square$

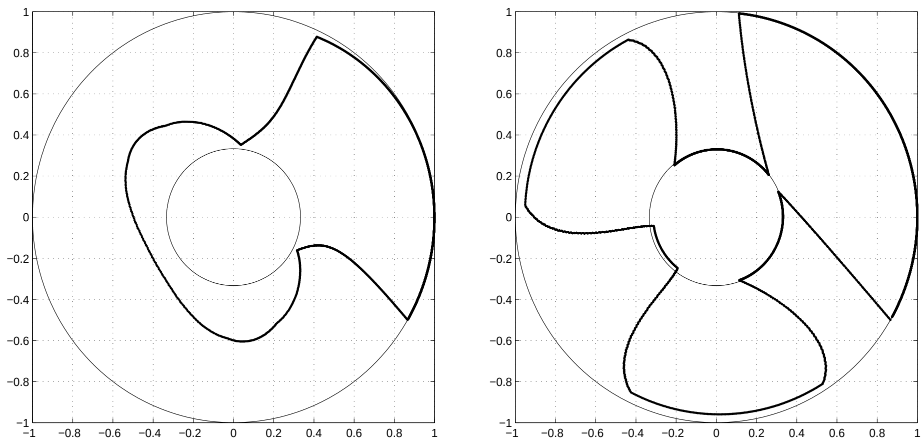
We illustrate the theorem on the following piece-wise continuous symbol which has six jump points,

$$a(\theta)=\begin{cases}\exp i\left[-\frac{\pi}{6}+\frac{2\pi}{3}\cdot\frac{7\theta}{\pi}\right], & \theta\in\left[0,\frac{\pi}{7}\right)\\ \frac{1}{3}\exp i\left[\frac{\pi}{6}+\frac{2\pi}{3}\cdot\left(\frac{7\theta}{\pi}-1\right)\right], & \theta\in\left[\frac{\pi}{7},\frac{2\pi}{7}\right)\\ \exp i\left[-\frac{\pi}{6}+\frac{2\pi}{3}\cdot\left(\frac{7\theta}{\pi}-2\right)\right], & \theta\in\left[\frac{2\pi}{7},\frac{3\pi}{7}\right)\\ \frac{1}{3}\exp i\left[-\frac{\pi}{6}+\frac{2\pi}{3}\cdot\left(\frac{7\theta}{\pi}-3\right)\right], & \theta\in\left[\frac{3\pi}{7},\frac{4\pi}{7}\right)\\ \exp i\left[-\frac{\pi}{6}+\frac{2\pi}{3}\cdot\left(\frac{7\theta}{\pi}-4\right)\right], & \theta\in\left[\frac{4\pi}{7},\frac{5\pi}{7}\right)\\ \frac{1}{3}\exp i\left[-\frac{\pi}{6}+\frac{2\pi}{3}\cdot\left(\frac{7\theta}{\pi}-5\right)\right], & \theta\in\left[\frac{5\pi}{7},\frac{6\pi}{7}\right)\\ \exp\left(-i\frac{\pi}{6}\right), & \theta\in\left[\frac{6\pi}{7},\pi\right]\end{cases}.$$

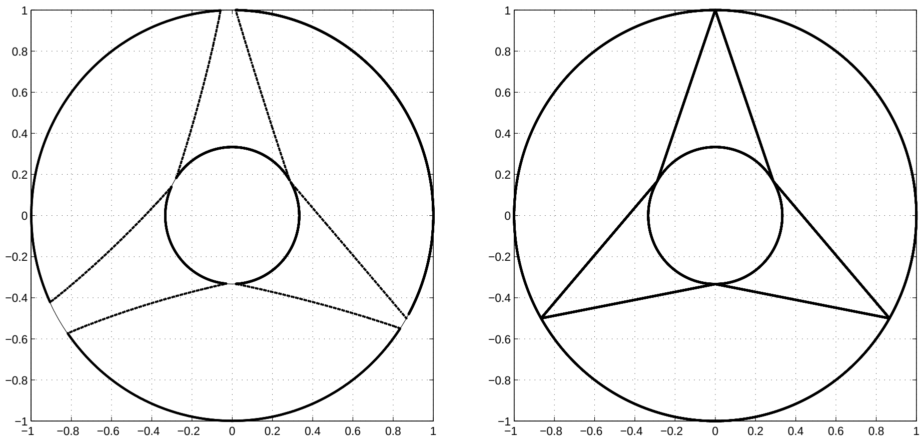
We show the image of the symbol $a=a(\theta)$, the image of $\gamma_{a,\lambda}$ for the following values of λ: 1, 10, 70, and 500, as well as the limit set $M_\infty(a)$.



The symbol $a(\theta)$ and the function $\gamma_{a,\lambda}$ for $\lambda=1$.



The function $\gamma_{a,\lambda}$ for $\lambda=10$ and $\lambda=70$.



The function $\gamma_{a,\lambda}$ for $\lambda = 500$ and the limit set $M_\infty(a)$.

We have obviously

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) \subset \operatorname{conv}(\operatorname{ess\,Range} a). \tag{5.11}$$

To illustrate the possible interrelations among these sets we can repeat the arguments of Examples 5.3 - 5.6 in [5] and construct the (piecewise continuous) symbols $a = a(\theta)$ to realize the following possibilities:

$$\begin{aligned} M_\infty(a) &= \operatorname{Range} a \quad (= \operatorname{ess\,Range} a), \\ M_\infty(a) &= \operatorname{conv}(\operatorname{ess\,Range} a) \quad (= \operatorname{conv}(\operatorname{Range} a)), \\ M_\infty(a) &\subset \partial \operatorname{conv}(\operatorname{Range} a), \\ M_\infty(a) &= \partial \operatorname{conv}(\operatorname{Range} a). \end{aligned}$$

(5.3) Unbounded symbols.

THEOREM (5.12). *Let $a(\theta) \in L_1(0,\pi) \cap C(0,1)$. Then*

$$\operatorname{Range} a \subset M_\infty(a).$$

Proof. We apply the Laplace method as in Theorem 5.2. Fix any point $\varphi \in (0,\pi)$ and consider for each ξ large enough the value $\lambda = 2\xi \arctan\varphi$. Then by (5.4) we have

$$\gamma_{a,\lambda}(\xi) = a(\varphi)(1 + \alpha(\lambda\sqrt{1 + (2\arctan\varphi)^{-2}})$$

where $\lim_{L\to\infty} \alpha(L) = 0$. Thus if $\xi \to \infty$ then $\lambda \to \infty$ as well and we have

$$a(\varphi) \in M_\infty(a).$$

□

The next theorem, whose proof is analogous to that of Theorem 5.11 in [5], shows that the property (5.11), previously established for bounded symbols, remains valid for summable symbols.

THEOREM (5.13). *Let $a(\theta) \subset L_1(0,\pi)$. Then*

$$M_\infty(a) \subset \operatorname{conv}(\operatorname{ess\,Range} a).$$

Note that for functions $a(\theta) \in L_1(0,\pi) \cap C(0,\pi)$ Theorems 5.12 and 5.13 imply that

$$\operatorname{Range} a \subset M_\infty(a) \subset \operatorname{conv}(\operatorname{Range} a),$$

and we show that $\operatorname{Range} a$ can coincide with each of these extreme sets.

EXAMPLE (5.14). For each $j \in \mathbb{N}$ define $I_j = [j^{-1} - j^{-3}, j^{-1}]$ and let $\overline{\{\xi_j\}_{j\in\mathbb{N}}} = [0, 2\pi]$. Define the symbol as follows

$$a(\theta) = \begin{cases} je^{i\xi_j}, & \theta \in I_j, \quad j \in \mathbf{N}; \\ 0, & \theta \in (0,\pi) \setminus \bigcup_{j=1}^\infty I_j. \end{cases}$$

It can be easily shown that

$$M_\infty(a) = \mathbb{C} = \operatorname{conv}(\operatorname{Range} a).$$

EXAMPLE (5.15). Given $\alpha \in [0,1)$, introduce $a(\theta) = (\sin\theta)^{i-\alpha}$, which is unbounded for $\alpha \in (0,1)$, but bounded and oscillating for $\alpha = 0$. Calculate using [4], formula 3.892.1,

$$\begin{aligned} \gamma_{a,\lambda}(\xi) &= \frac{\int_0^\pi (\sin\theta)^{\lambda+i-\alpha} e^{-2\xi\theta} d\theta}{\int_0^\pi (\sin\theta)^{\lambda} e^{-2\xi\theta} d\theta} \\ &= \frac{2^{\alpha-i}(\lambda+1)}{\lambda+i-\alpha+1} \frac{B\left(\frac{\lambda}{2}+1+i\xi, \frac{\lambda}{2}+1-i\xi\right)}{B\left(\frac{\lambda+i-\alpha}{2}+1+i\xi, \frac{\lambda+i-\alpha}{2}+1-i\xi\right)} \\ &= \frac{2^{\alpha-i}(\lambda+1)}{\lambda+i-\alpha+1} \frac{\Gamma(\lambda+2+i-\alpha)}{\Gamma(\lambda+2)} \frac{\Gamma\left(\frac{\lambda}{2}+1+i\xi\right)}{\Gamma\left(\frac{\lambda+i-\alpha}{2}+1+i\xi\right)} \\ &\times \frac{\Gamma\left(\frac{\lambda}{2}+1-i\xi\right)}{\Gamma\left(\frac{\lambda+i-\alpha}{2}+1-i\xi\right)}. \end{aligned}$$

Applying the asymptotic formulas for the Γ-function (see [4], formulas 8.327 and 8.328.2) we have

$$\gamma_{a,\lambda}(\xi) = \left[\left(\frac{(\lambda+2)^2}{(\lambda+2)^2+4\xi^2}\right)^{\frac{1}{2}}\right]^{i-\alpha} \left(1 + O\left(\frac{1}{\lambda+1}\right)\right).$$

Given any $v \in (0,\pi)$, we can take ξ and λ such that

$$\left(\frac{(\lambda+2)^2}{(\lambda+2)^2+4\xi^2}\right)^{\frac{1}{2}} = \sin v.$$

Thus

$$\gamma_{a,\lambda}(\xi) = (\sin v)^{i-\alpha} \left(1 + O\left(\frac{1}{\lambda+1}\right)\right),$$

and in this case $M_\infty(a) = \operatorname{Range} a$.

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