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Poly-Bergman spaces and two-dimensional singular integral operators

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Abstract. We describe a direct and transparent connection between the poly-Bergman type spaces on the upper half-plane and certain two dimensional singular integral operators.

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1. Introduction

We show that there is a direct and transparent connection between the poly-Bergman type spaces and certain two dimensional singular integral operators.

Recall that the poly-Bergman spaces $\mathcal{A}_n^2(\Pi)$ and $\widetilde{\mathcal{A}}_n^2(\Pi)$ on the upper half-plane Π, of analytic and anti-analytic functions respectively, are defined as the subspaces of $L_2(\Pi)$, endowed with the standard Lebesgue plane measure $dv(z) = dxdy$, $z = x + iy$, and consist of functions satisfying the following equations

$$\left(\frac{\partial}{\partial \overline{z}}\right)^n \varphi = \frac{1}{2^n} \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n \varphi = 0, \qquad n \in \mathbb{N},$$

and

$$\left(\frac{\partial}{\partial z}\right)^n \varphi = \frac{1}{2^n} \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^n \varphi = 0, \qquad n \in \mathbb{N},$$

respectively.

We introduce as well the following singular integral operators bounded on $L_2(\Pi)$:

$$(S_\Pi \varphi)(z) = -\frac{1}{\pi} \int_\Pi \frac{\varphi(\zeta)}{(\zeta - z)^2}\, dv(\zeta)$$

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and its adjoint

$$(S_\Pi^*\varphi)(z) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\zeta)}{(\overline{\zeta}-\overline{z})^2}\,dv(\zeta).$$

A. Dzhuraev [4, 5] showed that for a bounded domain D with smooth boundary the orthogonal projections $B_{D,n}$ and $\widetilde{B}_{D,n}$ of $L_2(D)$ onto the spaces $\mathcal{A}_n^2(D)$ and $\widetilde{\mathcal{A}}_n^2(D)$, respectively, can be expressed in the form

$$B_{\Pi,n} = I - (S_D)^n(S_D^*)^n + K_n \qquad \text{and} \qquad \widetilde{B}_{\Pi,n} = I - (S_D^*)^n(S_D)^n + \widetilde{K}_n,$$

where K_n and $\widetilde{K}_n$ are compact operators. Recently J. Ramírez and I. Spitkovsky [8] proved that in the case of the upper half-plane Π the compact summands K_n and $\widetilde{K}_n$ in the above formulas are equal to zero. Using this result Yu. Karlovich and L. Pessoa [7] described the action of the operators S_Π and S_Π^* on the poly-Bergman spaces, obtaining the statements of Theorem 3.5 below.

In the paper we propose another, more direct and transparent, approach to the problem, which follows the ideas of [9, 10] and gives the precise information about the structure of S_Π and S_Π^*. In Section 2 we present necessary facts from [9, 10]. The core result of the paper is contained in Theorems 3.1 and 3.2 and gives a simple (functional) model for the operators S_Π and S_Π^*: *each of them is unitary equivalent to the direct sum of two unilateral shifts, forward and backward, both taken with the infinite multiplicity.* This fact permits us an easy access to the majority of the properties of these operators. The most important properties, in the context of the paper, are given by the subsequent Theorems 3.5 and 3.7.

2. Poly-Bergman spaces

Let Π be the upper half-plane in $\mathbb{C}$, consider the space $L_2(\Pi)$ endowed with the usual Lebesgue plane measure $dv(z) = dxdy$, $z = x + iy$. Denote by $\mathcal{A}^2(\Pi)$ its Bergman subspace, i.e., the subspace which consists of all functions analytic in Π. It is well known that the Bergman projection B_Π of $L_2(\Pi)$ onto $\mathcal{A}^2(\Pi)$ has the form

$$(B_\Pi\varphi)(z) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\zeta)}{(z-\overline{\zeta})^2}\,dv(\zeta).$$

In addition to the Bergman space $\mathcal{A}^2(\Pi)$ introduce the space $\widetilde{\mathcal{A}}^2(\Pi)$ as the subspace of $L_2(\Pi)$ consisting of all functions anti-analytic in Π.

Further, analogously to the Bergman spaces $\mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}^2(\Pi)$, introduce the spaces of poly-analytic and poly-anti-analytic functions (see, for example, [1, 2, 4, 5]), the poly-Bergman spaces.

We define the space $\mathcal{A}_n^2(\Pi)$ of n-analytic functions as the subspace of $L_2(\Pi)$ of all functions $\varphi = \varphi(z,\overline{z}) = \varphi(x,y)$, which satisfy the equation

$$\left(\frac{\partial}{\partial\overline{z}}\right)^n\varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n\varphi = 0.$$

Similarly, we define the space $\widetilde{\mathcal{A}}_n^2(\Pi)$ of n-anti-analytic functions as the subspace of $L_2(\Pi)$ of all functions $\varphi = \varphi(z,\overline{z}) = \varphi(x,y)$, which satisfy the equation

$$\left(\frac{\partial}{\partial z}\right)^n \varphi = \frac{1}{2^n}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^n \varphi = 0.$$

Of course, we have $\mathcal{A}_1^2(\Pi) = \mathcal{A}^2(\Pi)$ and $\widetilde{\mathcal{A}}_1^2(\Pi) = \widetilde{\mathcal{A}}^2(\Pi)$, for $n = 1$, as well as $\mathcal{A}_n^2(\Pi) \subset \mathcal{A}_{n+1}^2(\Pi)$ and $\widetilde{\mathcal{A}}_n^2(\Pi) \subset \widetilde{\mathcal{A}}_{n+1}^2(\Pi)$, for each $n \in \mathbb{N}$.

Finally introduce the space $\mathcal{A}_{(n)}^2(\Pi)$ of true-n-analytic functions by

$$\mathcal{A}_{(n)}^2(\Pi) = \mathcal{A}_n^2(\Pi) \ominus \mathcal{A}_{n-1}^2(\Pi),$$

for $n > 1$, and by $\mathcal{A}_{(1)}^2(\Pi) = \mathcal{A}_1^2(\Pi)$; and, symmetrically, introduce the space $\widetilde{\mathcal{A}}_{(n)}^2(\Pi)$ of true-n-anti-analytic functions by

$$\widetilde{\mathcal{A}}_{(n)}^2(\Pi) = \widetilde{\mathcal{A}}_n^2(\Pi) \ominus \widetilde{\mathcal{A}}_{n-1}^2(\Pi),$$

for $n > 1$, and by $\widetilde{\mathcal{A}}_{(1)}^2(\Pi) = \widetilde{\mathcal{A}}_1^2(\Pi)$, for $n = 1$.

We have, of course,

$$\mathcal{A}_n^2(\Pi) = \bigoplus_{k=1}^n \mathcal{A}_{(k)}^2(\Pi) \qquad \text{and} \qquad \widetilde{\mathcal{A}}_n^2(\Pi) = \bigoplus_{k=1}^n \widetilde{\mathcal{A}}_{(k)}^2(\Pi).$$

To formulate the main result of this section we need more definitions. We start by introducing two unitary operators. Define the unitary operator

$$U_1 = F \otimes I : \; L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+), \tag{1}$$

where the Fourier transform $F : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is given by

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-ix\xi} f(\xi)\, d\xi. \tag{2}$$

The second unitary operator

$$U_2 : L_2(\Pi) = L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

is given by

$$(U_2\varphi)(x,y) = \frac{1}{\sqrt{2|x|}}\, \varphi(x, \frac{y}{2|x|}). \tag{3}$$

Then the inverse operator $U_2^{-1} = U_2^* : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$ acts as follows,

$$(U_2^{-1}\varphi)(x,y) = \sqrt{2|x|}\, \varphi(x, 2|x| \cdot y).$$

Recall (see, for example, [3]), that the Laguerre polinomial $L_n(y)$ of degree n, $n = 0,1,2,...$, and type 0 is defined by

$$\begin{aligned} L_n(y) \;&=\; L_n^0(y) = \frac{e^y}{n!}\,\frac{d^n}{dy^n}(e^{-y}\,y^n) \\ &=\; \sum_{k=0}^n \frac{n!}{k!(n-k)!}\,\frac{(-y)^k}{k!}, \qquad y \in \mathbb{R}_+, \end{aligned} \tag{4}$$

and that the system of functions

$$\ell_n(y) = e^{-y/2} L_n(y), \qquad n = 0, 1, 2, ... \tag{5}$$

forms an orthonormal basis in the space $L_2(\mathbb{R}_+)$.

Denote by L_n, $n = 0, 1, 2, ...$, the one-dimensional subspace of $L_2(\mathbb{R}_+)$ generated by the function $\ell_n(y)$.

The main result of the section reads as follows.

Theorem 2.1. *The unitary operator*

$$U = U_2 U_1 : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \to L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+)$$

provides the following isometrical isomorphisms of the above spaces:

1. *Isomorphic images of poly-analytic spaces*

$$\begin{aligned} U &: \mathcal{A}^2_{(n)}(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_{n-1}, \\ U &: \mathcal{A}^2_{n}(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes \bigoplus_{k=0}^{n-1} L_k, \\ U &: \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi) \longrightarrow L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}_+). \end{aligned}$$

2. *Isomorphic images of poly-anti-analytic spaces*

$$\begin{aligned} U &: \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \longrightarrow L_2(\mathbb{R}_-) \otimes L_{n-1}, \\ U &: \widetilde{\mathcal{A}}^2_{n}(\Pi) \longrightarrow L_2(\mathbb{R}_-) \otimes \bigoplus_{k=0}^{n-1} L_k, \\ U &: \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi) \longrightarrow L_2(\mathbb{R}_-) \otimes L_2(\mathbb{R}_+). \end{aligned}$$

3. *Furthermore we have the following decomposition of the space $L_2(\Pi)$*

$$\begin{aligned} L_2(\Pi) &= \bigoplus_{k=1}^{\infty} (\mathcal{A}^2_{(k)}(\Pi) \oplus \widetilde{\mathcal{A}}^2_{(k)}(\Pi)) \\ &= \bigoplus_{k=1}^{\infty} \mathcal{A}^2_{(k)}(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}^2_{(k)}(\Pi). \end{aligned}$$

3. Two-dimensional singular integral operators

We introduce the following singular integral operators bounded on $L_2(\Pi)$:

$$(S_\Pi \varphi)(z) = -\frac{1}{\pi} \int_\Pi \frac{\varphi(\zeta)}{(\zeta - z)^2} \, dv(\zeta)$$

and its adjoint

$$(S_\Pi^*\varphi)(z) = -\frac{1}{\pi}\int_\Pi \frac{\varphi(\zeta)}{(\overline{\zeta}-\overline{z})^2}\, dv(\zeta).$$

Note, that the operators S_Π and S_Π^* are the restrictions onto the upper half-plane Π of the following classical two-dimensional singular integral operators over $\mathbb{C} = \mathbb{R}^2$,

$$(S_{\mathbb{R}^2}\varphi)(z) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\varphi(\zeta)}{(\zeta - z)^2}\, dv(\zeta) \quad \text{and} \quad (S_{\mathbb{R}^2}^*\varphi)(z) = -\frac{1}{\pi}\int_{\mathbb{R}^2} \frac{\varphi(\zeta)}{(\overline{\zeta}-\overline{z})^2}\, dv(\zeta),$$

which are given in terms of the Fourier transform as follows,

$$S_{\mathbb{R}^2} = F^{-1}\frac{\overline{\zeta}}{\zeta}F \qquad \text{and} \qquad S_{\mathbb{R}^2}^* = S_{\mathbb{R}^2}^{-1} = F^{-1}\frac{\zeta}{\overline{\zeta}}F, \tag{6}$$

where $\zeta = \xi + i\eta = (\xi, \eta)$, and the Fourier transform F is given by

$$(F\varphi)(\zeta) = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{-i\zeta\cdot z}\,\varphi(z)\, dv(z),$$

where $z = x + iy = (x, y)$, and $\zeta \cdot z = \xi x + \eta y$.

By (6) these operators admit the following representations:

$$\begin{aligned} S_\Pi &= (I \otimes \chi_+ I) S_{\mathbb{R}^2}(I \otimes \chi_+ I) \\ &= (I \otimes \chi_+ I)(F^{-1} \otimes F^{-1})\frac{\xi - i\eta}{\xi + i\eta}(F \otimes F)(I \otimes \chi_+ I) \end{aligned} \tag{7}$$

and

$$\begin{aligned} S_\Pi^* &= (I \otimes \chi_+ I) S_{\mathbb{R}^2}^*(I \otimes \chi_+ I) \\ &= (I \otimes \chi_+ I)(F^{-1} \otimes F^{-1})\frac{\xi + i\eta}{\xi - i\eta}(F \otimes F)(I \otimes \chi_+ I), \end{aligned}$$

where ξ, $\eta \in \mathbb{R}$, and the one-dimensional Fourier transform F is given by (2).

Let us introduce the following integral operators

$$\begin{aligned} (S_+ f)(y) &= -f(y) + e^{-\frac{y}{2}}\int_0^y e^{\frac{t}{2}}\, f(t)\, dt, \\ (S_- f)(y) &= -f(y) + e^{\frac{y}{2}}\int_y^\infty e^{-\frac{t}{2}}\, f(t)\, dt, \end{aligned}$$

which, as we will see later on, are bounded on $L_2(\mathbb{R}_+)$ and are mutually adjoint.

As in Section 2 we will use the unitary operator

$$U = U_2U_1 : L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+) \longrightarrow L_2(\mathbb{R}) \otimes L_2(\mathbb{R}_+),$$

where the operators U_1 and U_2 are given by (1) and (3) respectively.

Theorem 3.1. *The unitary operator $U = U_2U_1$ gives an isometrical isomorphism of the space $L_2(\Pi) = [L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}_+)] \oplus [L_2(\mathbb{R}_-) \otimes L_2(\mathbb{R}_+)]$ under which the* *two-dimensional singular integral operators* S_Π *and* S_Π^* *are unitary equivalent to the following operators*

$$\begin{aligned} U\,S_\Pi\,U^{-1} &= (I\otimes S_+)\oplus(I\otimes S_-),\\ U\,S_\Pi^*\,U^{-1} &= (I\otimes S_-)\oplus(I\otimes S_+).\end{aligned}$$

Proof. By the representation (7) we have

$$\begin{aligned} S_1 &= U_1 S_\Pi U_1^{-1} = (F\otimes I)S_\Pi(F^{-1}\otimes I)\\ &= (I\otimes\chi_+ I)(I\otimes F^{-1})\frac{\xi-i\eta}{\xi+i\eta}(I\otimes F)(I\otimes\chi_+ I).\end{aligned}$$

The operator U_2 is unitary on both $L_2(\mathbb{R}_+)$ and $L_2(\mathbb{R})$, and furthermore it commutes with $\chi_{\mathbb{R}_+}I$. Direct calculation shows that

$$U_2(I\otimes F^{-1})\frac{\xi-i\eta}{\xi+i\eta}(I\otimes F)U_2^{-1} = (I\otimes F^{-1})\frac{\frac{1}{2}\mathrm{sign}x-i\eta}{\frac{1}{2}\mathrm{sign}x+i\eta}(I\otimes F).$$

Thus

$$\begin{aligned} S_2 &= US_\Pi U^{-1} = U_2 S_1 U_2^{-1}\\ &= (\chi_+ I\otimes\chi_+ I)(I\otimes F^{-1})\frac{\frac{1}{2}-i\eta}{\frac{1}{2}+i\eta}(I\otimes F)(\chi_+ I\otimes\chi_+ I)\\ &+ (\chi_- I\otimes\chi_+ I)(I\otimes F^{-1})\frac{\frac{1}{2}+i\eta}{\frac{1}{2}-i\eta}(I\otimes F)(\chi_- I\otimes\chi_+ I)\end{aligned}$$

and

$$\begin{aligned} S_2^* &= US_\Pi^* U^{-1}\\ &= (\chi_+ I\otimes\chi_+ I)(I\otimes F^{-1})\frac{\frac{1}{2}+i\eta}{\frac{1}{2}-i\eta}(I\otimes F)(\chi_+ I\otimes\chi_+ I)\\ &+ (\chi_- I\otimes\chi_+ I)(I\otimes F^{-1})\frac{\frac{1}{2}-i\eta}{\frac{1}{2}+i\eta}(I\otimes F)(\chi_- I\otimes\chi_+ I).\end{aligned}$$

The symbols of the two convolution operators

$$\widetilde{S}_+ = F^{-1}\frac{\frac{1}{2}-i\eta}{\frac{1}{2}+i\eta}F \quad\text{and}\quad \widetilde{S}_- = F^{-1}\frac{\frac{1}{2}+i\eta}{\frac{1}{2}-i\eta}F,$$

which are obviously bounded on $L_2(\mathbb{R})$, admit the following representations,

$$\frac{\frac{1}{2}-i\eta}{\frac{1}{2}+i\eta} = -1-\frac{i\eta}{\frac{1}{4}+\eta^2}+\frac{\frac{1}{2}}{\frac{1}{4}+\eta^2} \quad\text{and}\quad \frac{\frac{1}{2}+i\eta}{\frac{1}{2}-i\eta} = -1+\frac{i\eta}{\frac{1}{4}+\eta^2}+\frac{\frac{1}{2}}{\frac{1}{4}+\eta^2},$$

respectively.

Using the formulas 17.23.14 and 17.23.15 of [6] we have

$$F\left(\frac{\frac{1}{2}}{\frac{1}{4}+\eta^2}\right) = \sqrt{\frac{\pi}{2}}\,e^{-\frac{|y|}{2}}, \qquad F\left(\frac{i\eta}{\frac{1}{4}+\eta^2}\right) = \sqrt{\frac{\pi}{2}}\,\mathrm{sign}\,y\,e^{-\frac{|y|}{2}},$$

and thus

$$\begin{aligned}
(\widetilde{S}_+ f)(y) &= -f(y) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{\frac{\pi}{2}} e^{-\frac{|t-y|}{2}} (1 - \operatorname{sign}(t-y)) f(t) dt \\
&= -f(t) + \int_{\mathbb{R}} e^{-\frac{|t-y|}{2}} \chi_-(t-y) f(t) dt
\end{aligned}$$

and

$$\begin{aligned}
(\widetilde{S}_- f)(y) &= -f(y) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \sqrt{\frac{\pi}{2}} e^{-\frac{|t-y|}{2}} (1 + \operatorname{sign}(t-y)) f(t) dt \\
&= -f(t) + \int_{\mathbb{R}} e^{-\frac{|t-y|}{2}} \chi_+(t-y) f(t) dt.
\end{aligned}$$

Then the operators $S_+ = \chi_+ \widetilde{S}_+ \chi_+ I|_{L_2(\mathbb{R}_+)}$ and $S_- = \chi_+ \widetilde{S}_- \chi_+ I|_{L_2(\mathbb{R}_+)}$, acting on $L_2(\mathbb{R}_+)$, are as follows:

$$\begin{aligned}
(S_+ f)(y) &= -f(t) + \int_{\mathbb{R}_+} e^{-\frac{|t-y|}{2}} \chi_-(t-y) f(t) dt \\
&= -f(y) + e^{-\frac{y}{2}} \int_0^y e^{\frac{t}{2}} f(t) dt
\end{aligned}$$

and

$$\begin{aligned}
(S_- f)(y) &= -f(t) + \int_{\mathbb{R}_+} e^{-\frac{|t-y|}{2}} \chi_+(t-y) f(t) dt \\
&= -f(y) + e^{\frac{y}{2}} \int_y^\infty e^{-\frac{t}{2}} f(t) dt.
\end{aligned}$$

Thus finally

$$\begin{aligned}
U S_\Pi U^{-1} &= (\chi_+ I \otimes \chi_+ I)(I \otimes \widetilde{S}_+)(\chi_+ I \otimes \chi_+ I) \\
&+ (\chi_- I \otimes \chi_+ I)(I \otimes \widetilde{S}_-)(\chi_- I \otimes \chi_+ I) \\
&= \chi_+ I \otimes S_+ + \chi_- I \otimes S_- \\
&= (I \otimes S_+) \oplus (I \otimes S_-)
\end{aligned}$$

and

$$\begin{aligned}
U S_\Pi^* U^{-1} &= (\chi_+ I \otimes \chi_+ I)(I \otimes \widetilde{S}_-)(\chi_+ I \otimes \chi_+ I) \\
&+ (\chi_- I \otimes \chi_+ I)(I \otimes \widetilde{S}_+)(\chi_- I \otimes \chi_+ I) \\
&= \chi_+ I \otimes S_- + \chi_- I \otimes S_+ \\
&= (I \otimes S_-) \oplus (I \otimes S_+),
\end{aligned}$$

where the last lines in both representations are written according to the splitting

$$L_2(\Pi) = [L_2(\mathbb{R}_+) \otimes L_2(\mathbb{R}_+)] \oplus [L_2(\mathbb{R}_-) \otimes L_2(\mathbb{R}_+)] \,.$$

$\square$

We continue to use an orthonormal basis

$$\ell_n(y) = e^{-y/2} L_n(y), \qquad n = 0, 1, 2, ...,$$

of the space $L_2(\mathbb{R}_+)$, where the Lagguerre polynomials $L_n(y)$ are given by (4).

Theorem 3.2. *For each admissible n, the following equalities hold:*

$$(S_+\ell_n)(y) = -\ell_{n+1}(y), \qquad (S_-\ell_n)(y) = -\ell_{n-1}(y), \qquad \text{and} \qquad (S_-\ell_0)(y) = 0.$$

Proof. By [6], formula 8.971.1, we have

$$L'_n(y) - L'_{n+1}(y) = L_n(y). \tag{8}$$

Taking into account that $L_n(0) = 1$, for all n, the integral form of the above formula is as follows:

$$L_n(y) - L_{n+1}(y) = \int_0^y L_n(t)dt.$$

Calculate now

$$\begin{aligned}(S_+\ell_n)(y) &= -e^{-\frac{y}{2}} L_n(y) + e^{-\frac{y}{2}} \int_0^y L_n(t)dt \\ &= e^{-\frac{y}{2}} \left(-L_n(y) + L_n(y) - L_{n+1}(y)\right) = -\ell_{n+1}(y).\end{aligned}$$

Integrating by parts twice and using (8), we have

$$\begin{aligned}\int_y^\infty e^{-t} L_n(t)dt &= e^{-y} L_n(y) + \int_y^\infty e^{-t} L'_{n-1}(t)dt - \int_y^\infty e^{-t} L_{n-1}(t)dt \\ &= e^{-y} L_n(y) - \int_y^\infty e^{-t} L_{n-1}(t)dt \\ &- e^{-y} L_{n-1}(y) + \int_y^\infty e^{-t} L_{n-1}(t)dt \\ &= e^{-y} L_n(y) - e^{-y} L_{n-1}(y).\end{aligned}$$

Thus

$$\begin{aligned}(S_-\ell_n)(y) &= -e^{-\frac{y}{2}} L_n(y) + e^{\frac{y}{2}} \int_y^\infty e^{-t} L_n(t)dt \\ &= -e^{-\frac{y}{2}} L_n(y) + e^{\frac{y}{2}} \left(e^{-y} L_n(y) - e^{-y} L_{n-1}(y)\right) = -\ell_{n-1}(y).\end{aligned}$$

Finally,

$$(S_-\ell_0)(y) = -e^{-\frac{y}{2}} + e^{\frac{y}{2}} \int_y^\infty e^{-t} dt = 0.$$

□

It is convenient to change the previously used basis $\{\ell_n(y)\}_{n=0}^\infty$ of $L_2(\mathbb{R}_+)$ to the new basis $\{\widetilde{\ell}_n(y)\}_{n=0}^\infty$, where

$$\widetilde{\ell}_n(y) = (-1)^n \ell_n(y), \qquad n = 0, 1, 2, \ldots .$$

We note that the previously defined one-dimensional spaces L_n are generated by the new basis elements $\widetilde{\ell}_n(y)$ as well, and that the statements of Theorem 2.1 remain valid without any change.

Remark 3.3. *As the previous theorem shows, the operator S_+ is an isometric operator on $L_2(\mathbb{R}_+)$ and is nothing but the unilateral forward shift with respect to the basis $\{\widetilde{\ell}_n(y)\}_{n=0}^{\infty}$. Its adjoint operator S_- is the unilateral backward shift with respect to the same basis, and its kernel coincides with the one-dimensional space L_0 generated by $\widetilde{\ell}_0(y) = e^{-\frac{y}{2}}$.*

The above, together with Theorem 3.1, permits us to give a simple functional model for both operators S_Π and S_Π^*. Each of them is unitary equivalent to the direct sum of two unilateral shifts, forward and backward, both taken with the infinite multiplicity.

Let

$$L_n^{\oplus} = \bigoplus_{k=0}^{n} L_k$$

be the direct sum of the first $(n+1)$ L_k-spaces. We denote by P_n and $P_n^{\oplus}$ the orthogonal projections of $L_2(\mathbb{R}_+)$ onto L_n and $L_n^{\oplus}$, respectively.

Corollary 3.4. *For all admissible indices, we have*

$$\begin{aligned} P_0 &= I - S_+S_-, \\ P_n &= S_+^n P_0 S_-^n, \\ P_n^{\oplus} &= I - S_+^{n+1} S_-^{n+1}, \\ S_{+\,|L_n}^k &: L_n \longrightarrow L_{n+k}, \\ S_{-\,|L_n}^k &: L_n \longrightarrow L_{n-k}. \end{aligned}$$

The next result was obtained in [7] (see Theorem 2.4 and Corollary 2.6 therein) and shows that the action of both operators S_Π and S_Π^* is extremely transparent according to the decomposition

$$L_2(\Pi) = \bigoplus_{k=1}^{\infty} \mathcal{A}_{(k)}^2(\Pi) \oplus \bigoplus_{k=1}^{\infty} \widetilde{\mathcal{A}}_{(k)}^2(\Pi).$$

In our approach it is just a straightforward corollary of Theorems 2.1, 3.1, and Corollary 3.4.

Theorem 3.5. *For all admissible indices, we have*

$$\begin{array}{rcl}
(S_\Pi)^k\,|_{\mathcal{A}^2_{(n)}(\Pi)} & : & \mathcal{A}^2_{(n)}(\Pi) \longrightarrow \mathcal{A}^2_{(n+k)}(\Pi),\\
(S_\Pi)^k\,|_{\widetilde{\mathcal{A}}^2_{(n)}(\Pi)} & : & \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \longrightarrow \widetilde{\mathcal{A}}^2_{(n-k)}(\Pi),\\
(S^*_\Pi)^k\,|_{\widetilde{\mathcal{A}}^2_{(n)}(\Pi)} & : & \widetilde{\mathcal{A}}^2_{(n)}(\Pi) \longrightarrow \widetilde{\mathcal{A}}^2_{(n+k)}(\Pi),\\
(S^*_\Pi)^k\,|_{\mathcal{A}^2_{(n)}(\Pi)} & : & \mathcal{A}^2_{(n)}(\Pi) \longrightarrow \mathcal{A}^2_{(n-k)}(\Pi),
\end{array}$$

$$\ker(S_\Pi)^n = \widetilde{\mathcal{A}}^2_n(\Pi), \qquad (\mathrm{Im}\,(S_\Pi)^n)^\perp = \mathcal{A}^2_n(\Pi),$$
$$\ker(S^*_\Pi)^n = \mathcal{A}^2_n(\Pi), \qquad (\mathrm{Im}\,(S^*_\Pi)^n)^\perp = \widetilde{\mathcal{A}}^2_n(\Pi).$$

Corollary 3.6. *Each true-n-analytic function ψ admits the following representation,*

$$\psi = (S_\Pi)^{n-1}\varphi,$$

where $\varphi \in \mathcal{A}^2(\Pi)$.

Each true-n-anti-analytic function g admits the following representation,

$$g = (S^*_\Pi)^{n-1} f,$$

where $f \in \widetilde{\mathcal{A}}^2(\Pi)$.

We denote by $B_{\Pi,(n)}$ and $\widetilde{B}_{\Pi,(n)}$ the orthogonal projections of $L_2(\Pi)$ onto the spaces $\mathcal{A}^2_{(n)}(\Pi)$ and $\widetilde{\mathcal{A}}^2_{(n)}(\Pi)$, consisting of true-$n$-analytic and true-$n$-anti-analytic functions respectively. Let $B_{\Pi,n}$ and $\widetilde{B}_{\Pi,n}$ be the orthogonal projections of $L_2(\Pi)$ onto the spaces $\mathcal{A}^2_n(\Pi)$ and $\widetilde{\mathcal{A}}^2_n(\Pi)$, consisting of n-analytic and n-anti-analytic functions respectively.

We summarize now some important properties of the above projections in terms of singular operators.

Theorem 3.7. *For all admissible indices, we have*

$$\begin{array}{rcl}
B_\Pi & = & I - S_\Pi S^*_\Pi,\\
\widetilde{B}_\Pi & = & I - S^*_\Pi S_\Pi,\\
B_{\Pi,n} & = & I - (S_\Pi)^n (S^*_\Pi)^n,\\
\widetilde{B}_{\Pi,n} & = & I - (S^*_\Pi)^n (S_\Pi)^n,\\
B_{\Pi,(n)} = (S_\Pi)^{n-1} B_\Pi (S^*_\Pi)^{n-1} & = & (S_\Pi)^{n-1}(S^*_\Pi)^{n-1} - (S_\Pi)^n (S^*_\Pi)^n,\\
\widetilde{B}_{\Pi,(n)} = (S^*_\Pi)^{n-1} \widetilde{B}_\Pi (S_\Pi)^{n-1} & = & (S^*_\Pi)^{n-1}(S_\Pi)^{n-1} - (S^*_\Pi)^n (S_\Pi)^n,\\
B_{\Pi,(n+1)} & = & S_\Pi B_{\Pi,(n)} S^*_\Pi,\\
\widetilde{B}_{\Pi,(n+1)} & = & S^*_\Pi \widetilde{B}_{\Pi,(n)} S_\Pi.
\end{array}$$

Proof. Follows directly from Theorems 2.1, 3.1, and Corollary 3.4. □

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