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Integral Equations and Operator Theory

Commutative Algebras of Toeplitz Operators on the Reinhardt Domains

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Abstract. Let D be a bounded logarithmically convex complete Reinhardt domain in $\mathbb{C}^n$ centered at the origin. Generalizing a result for the one-dimensional case of the unit disk, we prove that the C^*-algebra generated by Toeplitz operators with bounded measurable separately radial symbols (i.e., symbols depending only on $|z_1|, |z_2|, \ldots, |z_n|$) is commutative.

We show that the natural action of the n-dimensional torus $\mathbb{T}^n$ defines (on a certain open full measure subset of D) a foliation which carries a transverse Riemannian structure having distinguished geometric features. Its leaves are equidistant with respect to the Bergman metric, and the orthogonal complement to the tangent bundle of such leaves is integrable to a totally geodesic foliation. Furthermore, these two foliations are proved to be Lagrangian.

We specify then the obtained results for the unit ball.

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1. Introduction

A family of recently discovered commutative C^*-algebras of Toeplitz operators on the unit disk (see for details [12, 13]) can be classified as follows. Each pencil of hyperbolic geodesics determines a set of symbols consisting of functions which are constant on the corresponding cycles, the orthogonal trajectories to geodesics forming a pencil. The C^*-algebra generated by Toeplitz operators with such symbols turns out to be commutative. Moreover, these commutative properties do not depend at all on smoothness properties of symbols: the corresponding symbols can be merely measurable. The prime cause appears to be the geometric configuration of level lines of symbols. Further it has been proved in [4] that, assuming some

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natural conditions on the "richness" of the symbol set, the above symbol sets are the only possible which gnerate commutative Toeplitz operator algebras on each (commonly considered) weighted Bergman space on the unit disk.

Recall that there are three different types of pencils of hyperbolic geodesics: an elliptic pencil, which is formed by geodesics intersecting in a single point, a parabolic pencil, which is formed by parallel geodesics, and a hyperbolic pencil, which is formed by disjoint geodesics, i.e., by all geodesics orthogonal to a given one. Note, that in all cases the cycles are equidistant in the hyperbolic metric.

The model case for elliptic pencils is when the geodesics intersect at the origin. In this case the geodesics are diameters and the cycles are the concentric circles centered at the origin. All other elliptic pencils can be obtained from this model by means of Möbius transformations. The commutative Toeplitz C^*-algebra for the elliptic model case is generated by Toeplitz operators with radial symbols.

As proved in [3], the C^*-algebras generated by Toeplitz operators with radial symbols, acting on the weighted Bergman spaces over the unit ball $\mathbb{B}^n$, are commutative as well.

In the present paper, we consider a more deep and natural multidimensional analog of the elliptic model pencil on the unit disk. We study Toeplitz operators on weighted Bergman spaces over bounded Reinhardt domains in $\mathbb{C}^n$, and prove, in particular, that the C^*-algebra generated by Toeplitz operators with bounded measurable separately radial symbols (i.e., symbols depending only on $|z_1|$, $|z_2|$, $\ldots$, $|z_n|$) is commutative.

Note that this single result can be also obtained directly by just calculating the matrix elements $\langle T_a z^p, z^q \rangle$, but we deliberately follow a more general procedure used in all model cases on the unit disk (see, for example, in [12]). This permits us to construct an analog of the Bargman transform (the operator R restricted on the (weighted) Bergman space), obtain the decomposition of the Bergman projection by means of R^* and R, and prepare these operators for the future use.

The second important question treated in the paper is the understanding of an adequate geometric description which generalizes geodesics and cycles of the unit disk to a multidimensional case. Each complete bounded Reinhardt domain D in $\mathbb{C}^n$ centered at the origin admits a natural action of the n-dimensional torus $\mathbb{T}^n$, and this action is isometric with respect to the Bergman metric in D. On a certain open full measure subset of D this action defines a foliation whose leaves are all diffeomorphic to $\mathbb{T}^n$. Furthermore, such foliation carries a transverse Riemannian structure having distinguished geometric features. First, the leaves are equidistant with respect to the Bergman metric, and second, the direction perpendicular to the leaves is totally geodesic, every geodesic which starts in the perpendicular direction to a leaf stays perpendicular to all other leaves. Now geometrically: the C^*-algebra generated by Toeplitz operators with bounded measurable symbols, which are constant on the leaves of the above foliation, is commutative. We prove also that the orthogonal complement to the tangent bundle of the $\mathbb{T}^n$-orbits is integrable, thus providing a pair of natural orthogonal foliations to a Reinhardt domain. Moreover, it turns out that both foliations are Lagrangian.

It is worth mentioning that the above geometric properties hold for each pencil of geodesics on the unit disk, but do not hold, for example, for the case of Toeplitz operators on the unit ball with radial symbols (the corresponding foliations are not Lagrangian).

We show that the unit ball $\mathbb{B}^n$ in $\mathbb{C}^n$ is the only Reinhardt domain which is, at the same time, bounded symmetric and irreducible. Then, we provide a detailed description of the extrinsic geometry of the foliation by $\mathbb{T}^n$-orbits in the unit ball. In particular, it is shown, through a computation of its second fundamental form, that certain geodesics in this foliation have geodesic curvatures with the same behavior found in the elliptic pencil of the unit disk. We then consider one-parameter families of weighted Bergman spaces in $\mathbb{B}^n$, commonly used in operator theory, and specify the results obtained to this special case.

Finally, using C. Fefferman's expression for the Bergman kernel of strictly pseudoconvex domains, we show that for any bounded complete Reinhardt domain with such pseudoconvexity property, the extrinsic curvature of the foliation coming from the $\mathbb{T}^n$-action has the same asymptotic behavior at (suitable) boundary points as the one observed in the unit ball.

2. Bergman space on the Reinhardt domains

Denote by $P(0,r)$, where $r = (r_1, \dots, r_n)$ and each $r_k > 0$, the closed polydisk in $\mathbb{C}^n$ centered at the origin:

$$P(0,r) = \{z = (z_1, \dots, z_n) : \ |z_k| \leq r_k, \ k = 1, \dots, n\}.$$

Recall (see, for example, [11]) that an open domain D in $\mathbb{C}^n$ is called the complete Reinhardt domain centered at the origin if for every its point z the polydisk $P(0, \tau(z))$, where $\tau(z) = (|z_1|, \dots, |z_n|)$, belongs to D. The set

$$\tau(D) = \{r = (r_1, \dots, r_n) = (|z_1|, \dots, |z_n|) \ : \ z = (z_1, \dots, z_n) \in D\},$$

which belongs to $\mathbb{R}^n_+ = \mathbb{R}_+ \times \cdots \times \mathbb{R}_+$, is called the base of the Reinhardt domain D.

The Reinhardt domain D is called logarithmically convex if the set $\log \tau(D)$ is convex. It is well known (see, for example, [11]) that the Reinhardt domain is logarithmically convex if and only if it is a domain of holomorphy, or if and only if it is a region of convergence of a power series.

Let now D be a bounded logarithmically convex complete Reinhardt domain in $\mathbb{C}^n$ centered at the origin. Consider a positive measurable function (weight) $\mu(r) = \mu(r_1, \dots, r_n)$, $r \in \tau(D)$, such that

$$\int_D \mu(|z|) dv(z) = (2\pi)^n \int_{\tau(D)} \mu(r) r dr < \infty,$$

where $dv(z) = dx_1 dy_1 \cdots dx_n dy_n$ is the usual Lebesgue measure in $\mathbb{C}^n$, $|z| = (|z_1|, \dots, |z_n|)$, and $r dr = \prod_{k=1}^n r_k dr_k$. We assume as well that the weight-function

$\mu(r)$ is bounded in some neighborhood of the origin and does not vanish in this neighborhood.

Introduce the weighted Hilbert space $L_2(D,\mu)$ with the scalar product

$$\langle f,g\rangle = \int_D f(z)\,\overline{g(z)}\,\mu(|z|)\,dv(z),$$

and its subspace, the weighted Bergman space $\mathcal{A}^2_\mu(D)$, which consists of all functions analytic in D. We denote as well by $B_{D,\mu}$ the (orthogonal) Bergman projection of $L_2(D,\mu)$ onto $\mathcal{A}^2_\mu(D)$.

Passing to the polar coordinates $z_k = t_k r_k$, where $t_k \in \mathbb{T} = S^1$, $k = 1,\dots,n$, and under the identification

$$z = (z_1,\dots,z_n) = (t_1r_1,\dots,t_nr_n) = (t,r),$$

where $t = (t_1,\dots,t_n) \in \mathbb{T}^n = \mathbb{T}\times\cdots\times\mathbb{T}$, $r = (r_1,\dots,r_n) \in \tau(D)$, we have $D = \mathbb{T}^n \times \tau(D)$ and

$$dv(z) = \prod_{k=1}^n \frac{dt_k}{it_k} \prod_{k=1}^n r_k dr_k.$$

That is we have the following decomposition

$$L_2(D,\mu) = L_2(\mathbb{T}^n) \otimes L_2(\tau(D),\mu),$$

where

$$L_2(\mathbb{T}^n) = \bigotimes_{k=1}^n L_2(\mathbb{T}, \frac{dt_k}{it_k})$$

and the measure $d\mu$ in $L_2(\tau(D),\mu)$ is given by

$$d\mu = \mu(r_1,\dots,r_n) \prod_{k=1}^n r_k dr_k.$$

We note that the Bergman space $\mathcal{A}^2_\mu(D)$ can be alternatively defined as the (closed) subspace of $L_2(D,\mu)$ which consists of all functions satisfying the equations

$$\frac{\partial}{\partial \overline{z}_k}\varphi = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial y_k}\right)\varphi = 0, \qquad k = 1,\dots,n,$$

or, in the polar coordinates,

$$\frac{\partial}{\partial \overline{z}_k}\varphi = \frac{t_k}{2}\left(\frac{\partial}{\partial r_k} - \frac{t_k}{r_k}\frac{\partial}{\partial t_k}\right)\varphi = 0, \qquad k = 1,\dots,n.$$

Define the discrete Fourier transform $\mathcal{F} : L_2(\mathbb{T}) \to l_2 = l_2(\mathbb{Z})$ by

$$\mathcal{F} :\, f \longmapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t)\, t^{-n}\, \frac{dt}{it}, \quad n \in \mathbb{Z}.$$

The operator $\mathcal{F}$ is unitary and

$$\mathcal{F}^{-1} = \mathcal{F}^* :\, \{c_n\}_{n\in\mathbb{Z}} \longmapsto f = \frac{1}{\sqrt{2\pi}} \sum_{n\in\mathbb{Z}} c_n\, t^n.$$

It is easy to check (see, for example, [12], Subsection 4.1) that the operator

$$u = (\mathcal{F} \otimes I)\frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t}\right)(\mathcal{F}^{-1} \otimes I) : l_2 \otimes L_2((0,1), rdr) \longrightarrow l_2 \otimes L_2((0,1), rdr)$$

acts as follows

$$u \, : \, \{c_k(r)\}_{k\in\mathbb{Z}} \longmapsto \left\{\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{k-1}{r}\right) c_{k-1}(r)\right\}_{k\in\mathbb{Z}}.$$

Introduce the unitary operator

$$U = \mathcal{F}_{(n)} \otimes I \, : \, L_2(\mathbb{T}^n) \otimes L_2(\tau(D), \mu) \longrightarrow \, l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu),$$

where $\mathcal{F}_{(n)} = \mathcal{F} \otimes \cdots \otimes \mathcal{F}$. Then the image $\mathcal{A}_1^2 = U(\mathcal{A}_\mu^2(D))$ of the Bergman space is the closed subspace of $l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu)$ which consists of all sequences $\{c_p(r)\}_{p\in\mathbb{Z}^n}$, $r = (r_1, \ldots, r_n) \in \tau(D)$, satisfying the equations

$$\frac{1}{2}\left(\frac{\partial}{\partial r_k} - \frac{p_k}{r_k}\right) c_{(p_1,\ldots,p_k)}(r_1, \ldots, r_n) = 0, \qquad k = 1, \ldots, n.$$

These equations are easy to solve, and their general solutions have the form

$$c_p(r) = \alpha_p c_p r^p, \qquad p = (p_1, \ldots, p_n) \in \mathbb{Z}^n,$$

where $c_p \in \mathbb{C}$, $r^p = r_1^{p_1} \cdot \cdots \cdot r_n^{p_n}$, and $\alpha_p = \alpha_{|p|}$ ($|p| = (|p_1|, \ldots, |p_n|)$, in this occurrence) is given by

$$\begin{aligned} \alpha_p &= \left(\int_{\tau(D)} r^{2|p|}\, \mu(r)\, rdr\right)^{-\frac{1}{2}} \\ &= \left(\int_{\tau(D)} r_1^{2|p_1|} \cdot \cdots \cdot r_n^{2|p_n|}\, \mu(r_1, \ldots, r_n) \prod_{k=1}^n r_k dr_k\right)^{-\frac{1}{2}}. \end{aligned} \tag{2.1}$$

Recall that each function $c_p(r) = \alpha_p c_p r^p$ has to be in $L_2(\tau(D), \mu)$, which implies that $c_p = 0$ for each $p = (p_1, \ldots, p_n)$ such that at least one of $p_k < 0$, $k = 1, \ldots, n$.

That is the space $\mathcal{A}_1^2 \subset l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu)$ coincides with the space of all sequences

$$c_p(r) = \begin{cases} \alpha_p c_p r^p, & p \in \mathbb{Z}_+^n = \mathbb{Z}_+ \times \cdots \times \mathbb{Z}_+ \\ 0, & p \in \mathbb{Z}^n \setminus \mathbb{Z}_+^n \end{cases},$$

and furthermore

$$\|\{c_p(r)\}_{p\in\mathbb{Z}_+^n}\|_{l_2(\mathbb{Z}^n)\otimes L_2(\tau(D),\mu)} = \|\{c_p\}_{p\in\mathbb{Z}_+^n}\|_{l_2(\mathbb{Z}^n)}.$$

Introduce now the isometric embedding

$$R_0 \, : \, l_2(\mathbb{Z}_+^n) \longrightarrow l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu)$$

as follows

$$R_0 \, : \, \{c_p\}_{p\in\mathbb{Z}_+^n} \longmapsto c_p(r) = \begin{cases} \alpha_p c_p r^p, & p \in \mathbb{Z}_+^n \\ 0, & p \in \mathbb{Z}^n \setminus \mathbb{Z}_+^n \end{cases}.$$

Then the adjoint operator $R_0^* : l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu) \longrightarrow l_2(\mathbb{Z}^n)$ is defined by

$$R_0^* \,:\, \{f_p(r)\}_{p\in\mathbb{Z}^n} \longmapsto \left\{ \alpha_p \int_{\tau(D)} r^p \, f_p(r) \, \mu(r_1, \dots, r_n) \prod_{k=1}^{n} r_k dr_k \right\}_{p\in\mathbb{Z}_+^n},$$

and it is easy to check that

$$\begin{aligned} R_0^* R_0 = I \quad &: \quad l_2(\mathbb{Z}_+^n) \longrightarrow l_2(\mathbb{Z}_+^n), \\ R_0 R_0^* = P_1 \quad &: \quad l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu) \longrightarrow \mathcal{A}_1^2, \end{aligned}$$

where P_1 is the orthogonal projection of $l_2(\mathbb{Z}^n) \otimes L_2(\tau(D), \mu)$ onto $\mathcal{A}_1^2$.

Summarizing the above we have

Theorem 2.1. *The operator $R = R_0 U$ maps $L_2(D, \mu)$ onto $l_2(\mathbb{Z}_+^n)$, and the restriction*

$$R|_{\mathcal{A}_\mu^2(D)} \,:\, \mathcal{A}_\mu^2(D) \longrightarrow l_2(\mathbb{Z}_+^n)$$

is an isometric isomorphism.

The adjoint operator

$$R^* = U^* R_0 \,:\, l_2(\mathbb{Z}_+^n) \longrightarrow \mathcal{A}_\mu^2(D) \subset L_2(D, \mu)$$

is the isometrical isomorphism of $l_2(\mathbb{Z}_+^n)$ onto the subspace $\mathcal{A}_\mu^2(D)$ of $L_2(D, \mu)$.

Furthermore

$$\begin{aligned} RR^* = I \quad &: \quad l_2(\mathbb{Z}_+^n) \longrightarrow l_2(\mathbb{Z}_+^n), \\ R^* R = B_{D,\mu} \quad &: \quad L_2(D, \mu) \longrightarrow \mathcal{A}_\mu^2(D), \end{aligned}$$

where $B_{D,\mu}$ is the Bergman projection of $L_2(D, \mu)$ onto $\mathcal{A}_\mu^2(D)$.

Theorem 2.2. *The isometric isomorphism*

$$R^* = U^* R_0 : l_2(\mathbb{Z}_+^n) \longrightarrow \mathcal{A}_\mu^2(D)$$

is given by

$$R^* \,:\, \{c_p\}_{p\in\mathbb{Z}_+^n} \;\longmapsto\; (2\pi)^{-\frac{n}{2}} \sum_{p\in\mathbb{Z}_+^n} \alpha_p \, c_p \, z^p. \tag{2.2}$$

Proof. Calculate

$$\begin{aligned} R^* = U^* R_0 \quad &: \quad \{c_p\}_{p\in\mathbb{Z}_+^n} \longmapsto U^*(\{\alpha_p \, c_p \, r^p\}_{p\in\mathbb{Z}_+^n}) \\ &= \quad (2\pi)^{-\frac{n}{2}} \sum_{p\in\mathbb{Z}_+^n} \alpha_p \, c_p \, (tr)^p = (2\pi)^{-\frac{n}{2}} \sum_{p\in\mathbb{Z}_+^n} \alpha_p \, c_p \, z^p. \end{aligned}$$

□

Corollary 2.3. *The inverse isomorphism*

$$R : \mathcal{A}_\mu^2(D) \longrightarrow l_2(\mathbb{Z}_+^n)$$

is given by

$$R : \varphi(z) \;\longmapsto\; \left\{ (2\pi)^{-\frac{n}{2}} \, \alpha_p \int_D \varphi(z) \, \overline{z}^p \, \mu(|z|) \, dv(z) \right\}_{p\in\mathbb{Z}_+^n}. \tag{2.3}$$

3. Toeplitz operators with separately radial symbols

We will call a function $a(z)$, $z \in D$, *separately radial* if $a(z) = a(r) = a(r_1, \ldots, r_n)$, i.e., a depends only on the radial components of $z = (z_1, \ldots, z_n) = (t_1 r_1, \ldots, t_n r_n)$.

Theorem 3.1. *Let $a = a(r)$ be a bounded measurable separately radial function. Then the Toeplitz operator T_a acting on $\mathcal{A}^2_\mu(D)$ is unitary equivalent to the multiplication operator $\gamma_a I = R\,T_a R^*$ acting on $l_2(\mathbb{Z}^n_+)$, where R and R^* are given by* (2.3) *and* (2.2) *respectively. The sequence $\gamma_a = \{\gamma_a(p)\}_{p\in\mathbb{Z}^n_+}$ is as follows*

$$\gamma_a(p) = \alpha_p^2 \int_{\tau(D)} a(r)\, r^{2p}\, \mu(r_1, \ldots, r_n) \prod_{k=1}^n r_k dr_k, \qquad p \in \mathbb{Z}^n_+, \tag{3.1}$$

where α_p is given by (2.1).

Proof. The operator T_a is obviously unitary equivalent to the operator

$$\begin{aligned} R\,T_a\,R^* &= R\,B_{D,\mu} a B_{D,\mu} R^* = R(R^*R)a(R^*R)R^* \\ &= (RR^*)RaR^*(RR^*) = RaR^* \\ &= R_0^* U a(r) U^{-1} R_0 \\ &= R_0^*(\mathcal{F}_{(n)} \otimes I)a(r)(\mathcal{F}^{-1}_{(n)} \otimes I)R_0 \\ &= R_0^* a(r) R_0. \end{aligned}$$

Now

$$\begin{aligned} R_0^* a(r) R_0 \{c_p\}_{p\in\mathbb{Z}^n_+} &= R_0^* \left\{a(r)\, \alpha_p\, c_p\, r^p\right\}_{p\in\mathbb{Z}^n_+} \\ &= \left\{\alpha_p \int_{\tau(D)} r^p\, a(r)\, \alpha_p\, c_p\, r^p\, \mu(r_1, \ldots, r_n) \prod_{k=1}^n r_k dr_k \right\}_{p\in\mathbb{Z}^n_+} \\ &= \{\gamma_a(p) \cdot c_p\}_{p\in\mathbb{Z}^n_+}, \end{aligned}$$

where

$$\gamma_a(p) = \alpha_p^2 \int_{\tau(D)} a(r)\, r^{2p}\, \mu(r_1, \ldots, r_n) \prod_{k=1}^n r_k dr_k, \qquad p \in \mathbb{Z}^n_+. \qquad \square$$

It is easy to see that the system of functions $\{e_p\}_{p\in\mathbb{Z}^n_+}$, where $e_p(z) = (2\pi)^{-\frac{n}{2}}\, \alpha_p\, z^p$, forms an orthonormal base in $\mathcal{A}^2_\mu(D)$.

Corollary 3.2. *The Toeplitz operator T_a with bounded measurable separately radial symbol $a(r)$ is diagonal with respect to the above orthonormal base:*

$$T_a\, e_p = \gamma_a(p) \cdot e_p, \qquad p \in \mathbb{Z}^n_+. \tag{3.2}$$

We can easily extend the notion of the Toeplitz operator for measurable *unbounded* separately radial symbols. Indeed, given a symbol $a = a(r) \in L_1(\tau(D), \mu)$, we still have equality (3.2). Then the densely defined (on the finite linear combinations of the above base elements) Toeplitz operator can be extended to a bounded operator on a whole $\mathcal{A}^2_\mu(D)$ if and only if the sequence $\gamma_a = \{\gamma_a(p)\}_{p\in\mathbb{Z}^n_+}$ is bounded. That is we have

Corollary 3.3. *The Toeplitz operator T_a with separately radial symbol $a = a(r) \in L_1(\tau(D),\mu)$ is bounded on $\mathcal{A}^2_\mu(D)$ if and only if*

$$\gamma_a = \{\gamma_a(p)\}_{p\in\mathbb{Z}^n_+} \in l_\infty,$$

and

$$\|T_a\| = \sup_{p\in\mathbb{Z}^n_+} |\gamma_a(p)|.$$

The Toeplitz operator T_a is compact if and only if $\gamma_a \in c_0$ that is

$$\lim_{p\to\infty} \gamma_a(p) = 0.$$

The spectrum of the bounded Toeplitz operator T_a is given by

$$\operatorname{sp} T_a = \overline{\{\gamma_a(p) : \, p \in \mathbb{Z}^n_+\}},$$

and its essential spectrum $\operatorname{ess}-\operatorname{sp} T_a$ coincides with the set of all limit points of the sequence $\{\gamma_a(p)\}_{p\in\mathbb{Z}^n_+}$.

Corollary 3.4. *The C^*-algebra generated by Toeplitz operators with separately radial L_∞-symbols is commutative.*

4. Foliations, transverse Riemannian structures, and bundle-like metrics

In this section we will briefly summarize some notions of foliations and their geometry. We refer to [8] for further details.

A foliation on a manifold M is a partition of M into connected submanifolds of the same dimension that locally looks like a partition given by the fibers of a submersion. The local picture is given by considering foliated charts and the partition as a global object is obtained by imposing a compatibility condition between the foliated charts. We make more precise this notion through the following definitions.

Definition 4.1. On a smooth manifold M a codimension q foliated chart is a pair (φ, U) given by an open subset U of M and a smooth submersion $\varphi : U \to V$, where V is an open subset of $\mathbb{R}^q$. For a foliated chart (φ, U) the connected components of the fibers of φ are called the plaques of the foliated chart. Two codimension q foliated charts (φ_1, U_1) and (φ_2, U_2) are called compatible if there exists a diffeomorphism $\psi_{12} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ such that the following diagram commutes:

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \varphi_1(U_1 \cap U_2) \arrow[rr, "\psi_{12}"] & & \varphi_2(U_1 \cap U_2) \end{tikzcd} \tag{4.1}$$

A foliated atlas on a manifold M is a collection $\{(\varphi_\alpha, U_\alpha)\}_\alpha$ of foliated charts which are mutually compatible and that satisfy $M = \bigcup_\alpha U_\alpha$.

It is straightforward to check that the compatibility of two foliated charts (φ_1, U_1) and (φ_2, U_2) ensures that, when restricted to $U_1 \cap U_2$, both submersions φ_1 and φ_2 have the same plaques. This in turn implies that, for any given foliated atlas, the following is an equivalence relation in M.

$$x \sim y \iff \begin{array}{l} \text{there is a sequence of plaques } (P_k)_{k=0}^l \text{ for foliated charts} \\ (\varphi_k, U_k)_{k=0}^l \text{ of the foliated atlas, such that } x \in P_0,\ y \in P_l, \\ \text{and } P_{k-1} \cap P_k \neq \phi \text{ for every } k = 1, \dots, l \end{array}$$

We will refer to the latter as the equivalence relation of the foliated atlas. It is a simple matter to show that the equivalence classes are in fact submanifolds of M of dimension $\dim(M) - q$, where q is the (common) codimension of the foliated charts.

Definition 4.2. A foliation $\mathfrak{F}$ on a manifold M is a partition of M which can be described as the classes of the equivalence relation of a foliated atlas. The classes are called the leaves of the foliation.

Suppose that M is a manifold carrying a smooth foliation $\mathfrak{F}$. We will denote with $T\mathfrak{F}$ the vector subbundle of TM that consists of elements tangent to the leaves of $\mathfrak{F}$. We can consider the quotient bundle $TM/T\mathfrak{F}$ which we will denote by $T^t\mathfrak{F}$. The latter will be referred to as the transverse vector bundle of the foliation $\mathfrak{F}$. Since $T^t\mathfrak{F}$ is a smooth vector bundle, we can consider the associated linear frame bundle which we will denote with $L_T(\mathfrak{F})$. More precisely, we have as a set:

$$L_T(\mathfrak{F}) = \{A : \ A : \mathbb{R}^q \to T^t_x\mathfrak{F} = T_xM/T_x\mathfrak{F} \text{ is an isomorphism and } x \in M\},$$

where q is the codimension of $\mathfrak{F}$ in M. It is easily seen that $L_T(\mathfrak{F})$ is a principal fiber bundle with structure group $\mathrm{GL}_q(\mathbb{R})$, we refer to [8] for the details of the proof. The principal bundle $L_T(\mathfrak{F})$ is called the transverse frame bundle since it allows us to study the geometry transverse to the foliation $\mathfrak{F}$.

When studying the transverse geometry of a foliation $\mathfrak{F}$ it is useful to consider a certain natural foliation in $L_T(\mathfrak{F})$, which is defined as follows.

Suppose that for a foliation $\mathfrak{F}$ on a manifold M we choose a foliated atlas $\{(\varphi_\alpha, U_\alpha)\}_\alpha$ that determines the foliation as in Definition 4.2. For any foliated chart $(\varphi_\alpha, U_\alpha)$ and every $x \in U_\alpha$ we have a linear map $d(\varphi_\alpha)_x : T_xM \to \mathbb{R}^q$ whose kernel is $T_x\mathfrak{F}$. This induces a linear isomorphism $d(\varphi_\alpha)^t_x : T^t_x\mathfrak{F} = T_xM/T_x\mathfrak{F} \to \mathbb{R}^q$. The latter allows us to define the smooth map:

$$\begin{array}{rcl} \varphi_\alpha^{(1)} : L_T(\mathfrak{F}|_{U_\alpha}) & \to & L(V_\alpha) \\ A & \mapsto & d(\varphi_\alpha)^t_x \circ A, \end{array}$$

where $L_T(\mathfrak{F}|_{U_\alpha})$ is the open subset of $L_T(\mathfrak{F})$ given by inverse image of U_α under the natural projection $L_T(\mathfrak{F}) \to M$, A is mapped to x under such projection and V_α is the target of φ_α. Next we observe that, since V_α is open in $\mathbb{R}^q$, the manifold

$L(V_\alpha)$ is open in $\mathbb{R}^q \times \mathrm{GL}_q(\mathbb{R})$ and so it is open in $\mathbb{R}^{q+q^2}$ as well. Furthermore, from our choices it is easy to check that the commutative diagram (4.1) and the compatibility of charts in a foliated atlas induce a corresponding commutative diagram given by:

$$\begin{tikzcd}
 & L_T(\mathfrak{F}|_{U_1 \cap U_2}) \arrow[dl, "\varphi^{(1)}_{\alpha_1}"'] \arrow[dr, "\varphi^{(1)}_{\alpha_2}"] & \\
L(\varphi_{\alpha_1}(U_1 \cap U_2)) \arrow[rr, "\psi^{(1)}_{\alpha_1\alpha_2}"] & & L(\varphi_{\alpha_2}(U_1 \cap U_2))
\end{tikzcd}$$

where $\psi^{(1)}_{\alpha_1\alpha_2}$ is defined as above for the diffeomorphism $\psi_{\alpha_1\alpha_2}$ for which we have $\varphi_{\alpha_2} = \psi_{\alpha_1\alpha_2} \circ \varphi_{\alpha_1}$, as in diagram (4.1). This shows that the set $\{(\varphi^{(1)}_\alpha, L_T(\mathfrak{F}|_{U_\alpha}))\}_\alpha$ defines a foliated atlas. The corresponding foliation in $L_T(\mathfrak{F})$ is called the lifted foliation. We state without proof the following result which can be found in [8].

Theorem 4.3. *Let $\mathfrak{F}$ be a foliation on a smooth manifold M. Then, the natural projection $L_T(\mathfrak{F}) \to M$ maps the leaves of the lifted foliation of $L_T(\mathfrak{F})$ locally diffeomorphically onto the leaves of $\mathfrak{F}$.*

From its construction, the principal fiber bundle $L_T(\mathfrak{F}) \to M$ models some aspects of the transverse geometry of the foliation $\mathfrak{F}$. At the same time, Theorem 4.3 shows that the lifted foliation in $L_T(\mathfrak{F})$ is needed to fully capture the foliated nature of the transverse geometry of $\mathfrak{F}$. In order to define transverse geometric structures for a given foliation $\mathfrak{F}$ we consider now reductions of $L_T(\mathfrak{F})$ compatible with the lifted foliation. More precisely, we have the following definition which also introduces the notion of a Riemannian foliation.

Definition 4.4. Let M be a manifold carrying a smooth foliation $\mathfrak{F}$ of codimension q, and let H be a Lie subgroup of $\mathrm{GL}_q(\mathbb{R})$. A transverse geometric H-structure is a reduction Q of $L_T(\mathfrak{F})$ to the subgroup H which is saturated with respect to the lifted foliation, i.e. such that $Q \cap L \neq \phi$ implies $L \subset Q$ for every leaf L of the lifted foliation. A transverse geometric $O(q)$-structure is also called a transverse Riemannian structure. A foliation endowed with a transverse Riemannian structure is called a Riemannian foliation.

From the definition, it is easy to see that a transverse Riemannian structure defines a Riemannian metric on the bundle $TM/T\mathfrak{F} = T^t\mathfrak{F}$. However, a transverse Riemannian structure is more than a simple Riemannian metric on $T^t\mathfrak{F}$. By requiring the $O(q)$-reduction that defines a transverse Riemannian structure to be saturated with respect to the lifted foliation, as in Definition 4.4, we ensure the invariance of the metric as we move along the leaves in M. This is a well known property of Riemannian foliations whose further discussion can be found in [8] and other books on the subject. Here we observe that, since a Riemannian metric on a manifold defines a distance, the invariance of a transverse Riemannian structure as we move along the leaves can be interpreted as the leaves of the foliation in M to be equidistant while we move along them. Again, this sort of remark is well known in the theory of foliations and shows that a Riemannian foliation has a distinguished geometry. In particular, not every foliation admits a Riemannian structure, a standard example is given by the Reeb foliation of the sphere S^3 (see [8]).

A fundamental way to construct transverse Riemannian structures for a foliation is to consider suitable Riemannian metrics on the manifold that carries the foliation. To describe such construction we will need some additional notions.

Definition 4.5. Let $\mathfrak{F}$ be a smooth foliation on a manifold M. A vector field X on M is called foliate if for every vector field Y tangent to the leaves of $\mathfrak{F}$ the vector field $[X, Y]$ is tangent to the leaves as well.

From the previous definition, we observe that the set of foliate vector fields is the normalizer of the fields tangent to the leaves of $\mathfrak{F}$ in the Lie algebra of all vector fields on M.

Definition 4.6. Let $\mathfrak{F}$ be a smooth foliation on a manifold M. A Riemannian metric h in M is called bundle-like for the foliation $\mathfrak{F}$ if the real-valued function $h(X, Y)$ is constant along the leaves of $\mathfrak{F}$ for every pair of vector fields X, Y which are foliate and perpendicular to $T\mathfrak{F}$ with respect to h.

Suppose that h is a Riemannian metric on a manifold M and that $\mathfrak{F}$ is a foliation on M. Then, the canonical projection $TM \to T^t\mathfrak{F}$ allows us to induce a Riemannian metric on the bundle $T^t\mathfrak{F}$, which in turn provides an $O(q)$-reduction of the transverse frame bundle $L_T(\mathfrak{F})$ (where q is the codimension of $\mathfrak{F}$). Nevertheless, such reduction does not necessarily defines a transverse Riemannian structure. The next result states that bundle-like metrics are precisely those that define transverse Riemannian structures. The proof of this theorem can be found in [8].

Theorem 4.7. *Let M be a manifold carrying a smooth foliation $\mathfrak{F}$ of codimension q. For every Riemannian metric h on M, denote by $O_T(M, h)$ the $O(q)$-reduction of $L_T(\mathfrak{F})$ given by the Riemannian metric on $T^t(\mathfrak{F})$ coming from h and the natural projection $TM \to T^t\mathfrak{F}$. If h is a bundle-like metric, then $O_T(M, h)$ defines a transverse Riemannian structure on $\mathfrak{F}$. Conversely, for every transverse Riemannian structure given by a reduction Q as in Definition* 4.4*, there is a bundle-like metric h on M such that $Q = O_T(M, h)$.*

Based on this result, we give the following definition.

Definition 4.8. Let $\mathfrak{F}$ be a Riemannian foliation on a manifold M. We will say that a bundle-like metric h on M is compatible with the Riemannian foliation if $O_T(M, h)$ is the reduction which defines the corresponding transverse Riemannian structure.

A fundamental property of Riemannian foliations is that, with respect to compatible bundle-like metrics, geodesics which start perpendicular to a leaf of the foliation stay perpendicular to all leaves.

Theorem 4.9. *Let $\mathfrak{F}$ be a Riemannian foliation on a manifold M and let h be a compatible bundle-like metric. If γ is a geodesic of h such that $\gamma'(t_0) \in (T_{\gamma(t_0)}\mathfrak{F})^\perp$, for some t_0, then $\gamma'(t) \in (T_{\gamma(t)}\mathfrak{F})^\perp$ for every t.*

This theorem is fundamental in the theory of Riemannian foliations and its proof can be found in [8]. We can provide its geometric interpretation as follows. Let M, $\mathfrak{F}$ and h be as in Theorem 4.9, and denote with $T\mathfrak{F}^\perp$ the orthogonal complement of $T\mathfrak{F}$ in TM; in particular, $TM = T\mathfrak{F} \oplus T\mathfrak{F}^\perp$. Hence, Theorem 4.9 states that every geodesic with an initial velocity vector in $T\mathfrak{F}^\perp$ has velocity vector contained in $T\mathfrak{F}^\perp$ for all time.

In a sense, the above states that the orthogonal complement $T\mathfrak{F}^\perp$ contains all geodesics perpendicular to $T\mathfrak{F}$. If the codimension of $\mathfrak{F}$ is 1, then $T\mathfrak{F}^\perp$ is one-dimensional and it can be integrated to a smooth one-dimensional foliation $\mathfrak{F}^\perp$ whose leaves are perpendicular to those of $\mathfrak{F}$. In such case, Theorem 4.9 ensures that the leaves of $\mathfrak{F}^\perp$ are geodesics with respect to the bundle-like metric h.

If $\mathfrak{F}$ has codimension greater than 1, then we can still consider the possibility of $T\mathfrak{F}^\perp$ to be integrable, e.g. to satisfy the hypothesis of Frobenius theorem (see [14]). In such case, we do have a foliation $\mathfrak{F}^\perp$ whose leaves are orthogonal to those of $\mathfrak{F}$. Again, in this case, Theorem 4.9 implies that the leaves of $\mathfrak{F}^\perp$ are totally geodesic. At the same time, the vector bundle $T\mathfrak{F}^\perp$ is not always integrable. Nevertheless, the above discussion shows that $T\mathfrak{F}^\perp$ can be thought of as being totally geodesic from a broader viewpoint. Alternatively, we can say that, from a geometric point of view, the foliation $\mathfrak{F}$ is transversely totally geodesic.

It is worth mentioning that the integrability of the bundle $T\mathfrak{F}^\perp$ given by a Riemannian foliation and a bundle-like metric is not at all trivial and requires strong restrictions on the geometry of the foliation or its leaves. As an example, we refer to [10], where the integrability of the corresponding $T\mathfrak{F}^\perp$ is only obtained for leaves carrying a suitable nonpositively curved Riemannian metric. At the same time, we will prove in the following sections that the orthogonal complement to the tangent bundle of the $\mathbb{T}^n$-orbits in a Reinhardt domain is integrable, which will then imply the presence of strong geometric features on such domains.

5. Extrinsic geometry of foliations

For a submanifold of any Riemannian manifold one can measure the obstruction for the submanifold to be a totally geodesic in the ambient. This also measures the extrinsic curvature of the submanifold, which is determined by the particular embedding and not just the inherited metric. We now briefly discuss some well known methods to study this extrinsic curvature and refer to [7] and [9] for further details. For our purposes it will be convenient and natural to discuss these notions for foliations.

Let $\widehat{M}$ be a Riemannian manifold and $\mathfrak{F}$ be a foliation of $\widehat{M}$ having codimension q and with p-dimensional leaves. We will denote by $\widehat{\nabla}$ the Levi-Civita connection of $\widehat{M}$ and by ∇ the connection of the bundle $T\mathfrak{F}$ obtained by pasting together the Levi-Civita connections of the leaves of $\mathfrak{F}$ for the metric inherited from that of $\widehat{M}$. Let us also denote by $\mathcal{V}$ and $\mathcal{H}$ the orthogonal projections of $T\widehat{M}$ onto $T\mathfrak{F}$ and $T\mathfrak{F}^\perp$, respectively. These projections are respectively called the vertical and horizontal projections with respect to $\mathfrak{F}$. Then the following holds (see [7]):

Lemma 5.1. *The connection ∇ is the vertical projection of $\widehat{\nabla}$. More precisely, we have:*

$$\nabla_X Y = \mathcal{V}(\widehat{\nabla}_X Y),$$

for every pair of vector fields in $\widehat{M}$ everywhere tangent to the leaves of $\mathfrak{F}$.

We recall that the Levi-Civita connection is the differential operator that allows to define geodesics. Hence, the previous result shows that the obstruction for the leaves of $\mathfrak{F}$ to be totally geodesic is precisely the difference between $\widehat{\nabla}$ and its vertical projection as above, in other words, the horizontal projection of $\widehat{\nabla}$. This suggests to introduce the following classical definition (see [7]).

Definition 5.2. Let $\mathfrak{F}$ be a foliation of a Riemannian manifold $\widehat{M}$. The second fundamental form II of the leaves of $\mathfrak{F}$ is given at every $x \in \widehat{M}$ by:

$$\begin{aligned} \mathrm{II}_x : T_x\mathfrak{F} \times T_x\mathfrak{F} &\to T_x\mathfrak{F}^\perp \\ (u, v) &\mapsto \mathcal{H}(\widehat{\nabla}_X Y)_x, \end{aligned}$$

where X, Y are vector fields defined in a neighborhood of x in $\widehat{M}$ everywhere tangent to $\mathfrak{F}$ and such that $X_x = v$ and $Y_x = v$.

It is very well known that the definition of II_x as above does not depend on the choice of the vector fields X and Y. It is also known that the second fundamental form at every every point is a symmetric bilinear form that defines a tensor which is a section of the bundle $T\mathfrak{F}^* \otimes T\mathfrak{F}^* \otimes T\mathfrak{F}^\perp$.

As it occurs with any tensor, it is easier to describe some of the properties of II by introducing local bases for the bundles involved and computing the components with respect to such bases. This is particularly useful if one has global bases for the bundles. These are defined more precisely as follows.

Definition 5.3. Let $E \to \widehat{M}$ be a subbundle of the tangent bundle of the Riemannian manifold $\widehat{M}$. Then, a collection $(V_1, \dots, V_k)$ of sections of E defined on all of $\widehat{M}$ is called a global framing of E if for every $x \in \widehat{M}$, the set of tangent vectors $(V_1(x), \dots, V_k(x))$ is a basis for the fiber E_x.

The next result is an obvious consequence of the symmetry of II. It will allow us to simplify the computation of the values for II.

Proposition 5.4. *Let $\widehat{M}$ be a Riemannian manifold with a foliation $\mathfrak{F}$ as above. Suppose that $(V_k)_{k=1}^p$ is a global framing of $T\mathfrak{F}$. Then II as a tensor is completely determined by the vector fields $\mathrm{II}(V_k + V_l, V_k + V_l)$ for $k, l = 1, \dots, p$.*

Proof. It is enough to use the relation:

$$\mathrm{II}\,(V_k, V_l) = \frac{1}{2}(\mathrm{II}\,(V_k + V_l, V_k + V_l) - \mathrm{II}\,(V_k, V_k) - \mathrm{II}\,(V_l, V_l))$$

which is satisfied by the symmetry of II. □

6. Isometric actions of Lie groups

In this section we will consider some general notions about actions of Lie groups on a manifold preserving a Riemannian metric. In what follows M will denote a smooth manifold and G a connected Lie group acting smoothly on the left on M. For the next definition, we recall that the stabilizer of a point $x \in M$ for the G-action is the set $G_x = \{g \in G : gx = x\}$.

Definition 6.1. The action of G on M is called free (locally free) if for every $x \in M$ the stabilizer G_x is trivial (respectively discrete).

A straightforward application of Frobenius theorem on the integrability of vector subbundles of a tangent bundle (see [14]) allows us to obtain the following result.

Proposition 6.2. *If G acts locally freely on M, then the G-orbits define a smooth foliation on M.*

Proof. Denote by $\mathfrak{g}$ the Lie algebra of G. Then for every $X \in \mathfrak{g}$ we can define the transformations of M given by the maps:

$$\begin{array}{rcl} \varphi_t : M & \to & M \\ x & \mapsto & \exp(tX)x \end{array}$$

for every $t \in \mathbb{R}$. This family of maps is in fact a one-parameter group of diffeomorphism of M, in other words, we have:

$$\varphi_{t_1+t_2} = \varphi_{t_1} \circ \varphi_{t_2}$$

for every $t_1, t_2 \in \mathbb{R}$. Hence, there is a smooth vector field X^* on M given by:

$$X^*_x = \frac{d}{dt}\Big|_{t=0}(\exp(tX)x).$$

Also, it is easy to check that the global flow of X^* is given by $(\varphi_t)_t$. Furthermore, since the Lie group G acts locally freely, the condition $X^*_x = 0$ for some $x \in M$ implies $X = 0$; otherwise the subgroup $(\exp(tX))_t$ would be nondiscrete and contained in G_x for some $x \in M$.

From the above remarks it follows that the map:

$$\begin{array}{rcl} M \times \mathfrak{g} & \to & TM \\ (x, X) & \mapsto & X^*_x \end{array}$$

is a smooth vector bundle inclusion which thus defines a subbundle $T\mathcal{O}$ of TM.

On the other hand, by using the results in [6], the following relation holds for every $X, Y \in \mathfrak{g}$:

$$[X^*, Y^*] = -[X, Y]^*$$

From this it is easy to conclude that the smooth sections of $T\mathcal{O}$ are closed under the Lie brackets of smooth vector fields. By Frobenius theorem (see [14]) the vector subbundle $T\mathcal{O}$ induces a smooth foliation whose leaves have the fibers of $T\mathcal{O}$ as tangent spaces. Since G is connected it is generated by the set $\exp \mathfrak{g}$, and so one can conclude that the leaves of such foliation are precisely the G-orbits. □

In the proof of the previous result it is shown that the tangent bundle of the foliation by G-orbits is $T\mathcal{O}$. Whenever G acts locally freely we will use $T\mathcal{O}$ to denote such tangent bundle.

We will now consider the case where G acts locally freely preserving a Riemannian metric on M.

Theorem 6.3. *If G acts locally freely on M preserving a Riemannian metric h, then the G-orbits define a smooth Riemannian foliation for which h is a compatible bundle-like metric.*

Proof. By Theorem 4.7 it is enough to show that h is bundle-like with respect to the foliation by G-orbits given by Proposition 6.2.

Choose X and Y foliate vector fields perpendicular to the G-orbits. We need to prove that $v(h(X, Y)) = 0$, for every $v \in T\mathcal{O}$. By the proof of Proposition 6.2 there exists $Z \in \mathfrak{g}$, the Lie algebra of G, such that $Z^*_x = v$, where x is the basepoint of v. Hence, it suffices to prove that $Z^*(h(X, Y)) = 0$ for every $Z \in \mathfrak{g}$.

For any Z^* as above, we denote with L_{Z^*} the Lie derivative with respect to Z^* and refer to [6] for the definition. In fact, from [6] it follows that L_{Z^*} when applied to h yields a bilinear form that satisfies:

$$(L_{Z^*}h)(X, Y) = Z^*(h(X, Y)) - h([Z^*, X], Y) - h(X, [Z^*, Y]). \tag{6.1}$$

On the other hand, since the one-parameter group $(\exp(tX))_t$ acts by isometries on (M, h), i.e. preserving h, it follows that Z^* is a Killing field for h and so it satisfies:

$$(L_{Z^*}h)(X, Y) = 0, \tag{6.2}$$

we refer to [6] for this fact and the definitions involved. From equations (6.1) and (6.2) we obtain:

$$Z^*(h(X, Y)) = h([Z^*, X], Y) + h(X, [Z^*, Y]).$$

Then we observe that, since X and Y are foliate, the vector fields $[Z^*, X]$ and $[Z^*, Y]$ are tangent to the G-orbits, and so the terms on the right-hand side of the last equation vanish since X and Y are also perpendicular to the G-orbits. This shows that $Z^*(h(X, Y)) = 0$ thus concluding the proof. □

From the last result and Theorem 4.9 we obtain the following consequence.

Theorem 6.4. *If G acts locally freely on M preserving a Riemannian metric h and γ is a geodesic (with respect to h) perpendicular at some point to a G-orbit, then γ intersects every G-orbit perpendicularly.*

The previous result and the remarks following Theorem 4.9 allows us to say that, from a geometric point of view, every locally free action of a group G preserving a Riemannian metric h defines (through its orbits) a foliation which is transversely totally geodesic.

7. Lagrangian foliations associated with a Reinhardt domain

We now proceed to study the geometry of Reinhardt domains. For this we will obtain some properties of its Bergman metric and apply the foliation theory considered in the previous sections. As before, in this section $D \subset \mathbb{C}^n$ denotes a bounded logarithmically convex complete Reinhardt domain centered at the origin.

Using the monomial orthonormal base $\{e_p\}_{p\in\mathbb{Z}_+^n}$ of $\mathcal{A}_\mu^2(D)$, mentioned in Section 3, we have obviously

Lemma 7.1. *The Bergman kernel K_D of the domain D admits the following representation*

$$K_D(z,\zeta) = (2\pi)^{-n} \sum_{p\in\mathbb{Z}_+^n} \alpha_p^2\, z^p \bar{\zeta}^p,$$

where the coefficients α_p, $p \in \mathbb{Z}_+^n$, are given by (2.1). *In particular, the function $K_D(z,z)$ depends only on r.*

In this section we will use the polar coordinates $z_k = r_k t_k = r_k e^{i\theta_k}$, $k = 1,\dots,n$, for points $z = (z_1,\dots,z_n) \in D$.

Theorem 7.2. *Let ds_D^2 be the Bergman metric of D considered as a Hermitian metric and $h_D = \mathrm{Re}\,(ds_D^2)$ the associated Riemannian metric. Then:*

$$h_D = \sum_{k,l=1}^n F_{kl}(r)(dr_k \otimes dr_l + r_k r_l d\theta_k \otimes d\theta_l),$$

where the functions F_{kl} are given by:

$$F_{kl}(r) = \frac{1}{4}\left(\frac{\partial^2}{\partial r_k \partial r_l} + \frac{\delta_{kl}}{r_k}\frac{\partial}{\partial r_k}\right)\log K_D(z,z),$$

and depend only on r.

Proof. For the Bergman kernel K_D, the associated Bergman metric considered as a Hermitian metric is given by:

$$ds_D^2 = \sum_{k,l=1}^n \frac{\partial^2 \log K_D(z,z)}{\partial z^k \partial \bar{z}^l} dz^k \otimes d\bar{z}^l.$$

Let $F(r) = F(z) = \log K_D(z,z)$, which by Lemma 7.1 depends only on r. Then a straightforward computation shows that:

$$\frac{\partial^2 F}{\partial z_k \partial \bar{z}_l}(z) = \frac{1}{4}\left(\frac{\bar{z}_k z_l}{r_k r_l}\frac{\partial^2 F}{\partial r_k \partial r_l}(z) + \frac{\delta_{kl}}{r_k}\frac{\partial F}{\partial r_k}(z)\right).$$

The required identity is then obtained by replacing these expressions into that of ds_D^2, using the relations $z_k = r_k e^{i\theta_k}$ and computing the real part of the expression thus obtained. □

Consider the following action of the n-dimensional torus $\mathbb{T}^n$ on D

$$\begin{array}{rcl} \mathbb{T}^n \times D & \to & D \\ (t,z) & \mapsto & tz, \end{array}$$

which being biholomorphic yields the following immediate consequence.

Theorem 7.3. *Let h_D be the Riemannian metric of D defined by its Bergman metric. Then $\mathbb{T}^n$ acts isometrically on (D, h_D).*

Note that the action of $\mathbb{T}^n$ is not locally free at all points of an n-dimensional Reinhardt domain, but it is almost so as the following obvious result states. We recall that in a measure space, a subset is called conull if its complement has zero measure.

Lemma 7.4. *For D as before, the set:*

$$\widehat{D} = \{z \in D : \ z_k \neq 0 \text{ for every } k = 1, \dots, n\}$$

is the set of points whose stabilizers with respect to the action of $\mathbb{T}^n$ are discrete. Furthermore, $\widehat{D}$ is an open conull subset of D on which $\mathbb{T}^n$ acts freely.

As a consequence of Theorems 6.3 and 7.3 and Lemma 7.4 we obtain the following.

Theorem 7.5. *Let D be as before, $\widehat{D}$ the subset of D defined in Lemma 7.4 and h_D the Riemannian metric defined by the Bergman metric of D. Then, the $\mathbb{T}^n$-orbits in $\widehat{D}$ define a Riemannian foliation $\mathcal{O}$ for which h is a compatible bundle-like metric.*

Given such result we now obtain the following statement which makes use of Theorem 6.4 as well.

Theorem 7.6. *Let D be as before, $\widehat{D}$ the subset of D defined in Lemma 7.4 and h_D the Riemannian metric defined by the Bergman metric of D. If γ is a geodesic in $\widehat{D}$ (with respect to h) perpendicular at some point to a $\mathbb{T}^n$-orbit, then γ intersects every $\mathbb{T}^n$-orbit perpendicularly.*

We now prove that the Riemannian foliation $\mathcal{O}$ obtained in the previous result is Lagrangian.

Theorem 7.7. *Let D be as before, $\widehat{D}$ the subset of D defined in Lemma 7.4, ds^2_D the Bergman metric of D as a Hermitian metric and $\mathcal{O}$ the Riemannian foliation of $\mathbb{T}^n$-orbits in $\widehat{D}$. Then $\mathcal{O}$ is Lagrangian with respect to the Riemannian metric $h_D = \mathrm{Re}\,(ds^2_D)$, in other words, the $\mathbb{T}^n$-orbits in $\widehat{D}$ are Lagrangian with respect to h_D.*

Proof. We need to prove that $T_z\mathcal{O}$ and $iT_z\mathcal{O}$ are perpendicular with respect to the Riemannian metric $h_D = \mathrm{Re}\,(ds^2_D)$ at every $z \in \widehat{D}$. Since such condition is invariant under the $\mathbb{T}^n$-action we can assume that $z = x \in \mathbb{R}^n_+$.

We observe that for every $x \in \widehat{D}$ we have $T_x\mathcal{O} = i\mathbb{R}^n$. Hence the result follows by using Theorem 7.2 together with the fact that $i\mathbb{R}^n$ and $\mathbb{R}^n$ are perpendicular with respect to the elements $dr_k \otimes dr_l + r_k r_l d\theta_k \otimes d\theta_l$ for every k, l. □

We now prove that the normal bundle to $\mathcal{O}$ is integrable.

Theorem 7.8. *Let D be as before, $\widehat{D}$ the subset of D defined in Lemma 7.4 and h_D the Riemannian metric defined by the Bergman metric of D. If we denote with $T\mathcal{O}^\perp$ the vector subbundle of $T\widehat{D}$ of tangent vectors perpendicular to $\mathcal{O}$, then $T\mathcal{O}^\perp$ is integrable to a foliation $\mathcal{P}$. Furthermore, $\mathcal{P}$ is a Lagrangian totally geodesic foliation of $\widehat{D}$.*

Proof. If we let $M_0 = D \cap \mathbb{R}^n_+$, then by the proof of Theorem 7.7 the tangent bundle to M_0 coincides with $T\mathcal{O}^\perp$ restricted to M_0, and so M_0 is an integral submanifold of $T\mathcal{O}^\perp$. Since $T\mathcal{O}^\perp$ is invariant under the $\mathbb{T}^n$-action and such action preserves the metric, it follows that for every $t \in \mathbb{T}^n$ the manifold:

$$M_t = tM_0$$

is an integral submanifold of $T\mathcal{O}^\perp$, thus showing the integrability of such bundle to some foliation $\mathcal{P}$.

By Lemma 7.1 we have $T\mathcal{P} = T\mathcal{O}^\perp = iT\mathcal{O}$ which implies that $\mathcal{P}$ is Lagrangian. Finally $\mathcal{P}$ is totally geodesic by Theorem 7.6. □

We now state the following easy corollary of the previous discussion.

Corollary 7.9. *The sets of vector fields:*

$$\left(\frac{\partial}{\partial\theta_k}\right)_{k=1}^n \qquad \text{and} \qquad \left(\frac{\partial}{\partial r_k}\right)_{k=1}^n,$$

define global framings for the bundles $T\mathcal{O}$ and $T\mathcal{O}^\perp = T\mathcal{P}$, respectively, on $\widehat{D}$.

8. The unit ball

An important class of domains in complex analysis is given by those which are bounded and symmetric. The next result shows that each irreducible bounded symmetric domain which is also Reinhardt has to be a unit ball. As usual, we will denote by $\mathbb{B}^n$ the unit ball in $\mathbb{C}^n$.

Theorem 8.1. *Let D be an irreducible bounded symmetric domain. Then D is also a Reinhardt domain if and only if $D = \mathbb{B}^n$ for some $n \in \mathbb{Z}_+$.*

Proof. First, the unit ball centered at the origin in a complex vector space is obviously a Reinhardt domain. Conversely, let us assume that D is an irreducible bounded symmetric domain which is also Reinhardt. We show that it is a unit ball centered at the origin of some complex vector space. For this we use Cartan's classification of irreducible bounded symmetric domains and the description of their biholomorphisms as found in [5]. We present the needed basic properties in Table 1, which recollects some of the information found in Table V in page 518 from [5]. Every irreducible bounded symmetric domain D in Table 1 is identified by its type in the first column (following the notation from [5]) and is explicitly given as the quotient G_0/K for the groups in the second and third column. The group G_0 is, up to a finite covering, the group of biholomorphisms of D and K is the subgroup of G_0 consisting of those transformations that fix the origin. For the exceptional bounded symmetric domains of type **EIII** and **EVII** we write down the Lie algebras of the corresponding groups, which is enough for our purposes; again, we follow here the notation from [5] to identify real forms of exceptional complex Lie algebras. The last two columns permit us to compare the complex dimension of D and the dimension of a maximal torus T in K. This last dimension is well known from the basic properties of the compact groups K that appear in Table 1. We recall from the basic theory of symmetric spaces that the universal covering of the group G_0 completely determines the bounded symmetric domain: in other words, two bounded symmetric domains whose corresponding groups G_0 in Table 1 have the same universal covering group are biholomorphic. Through out Table 1, the symbols p, q and n are assumed to be positive integers. The additional conditions on the types **BDI(2,q)** and **DIII** are required for the corresponding quotient G_0/K to actually define an irreducible bounded symmetric domain.

Table 1. Irreducible bounded symmetric domains

D	G_0	K	$\dim_{\mathbb{C}}(D)$	$\dim(T)$
AIII	$SU(p,q)$	$S(U(p) \times U(q))$	pq	$p+q-1$
BDI(2,q) ($q \neq 2$)	$SO_0(2,q)$	$SO(2) \times SO(q)$	q	$\left[\frac{q}{2}\right]+1$
DIII ($n \geq 2$)	$SO^*(2n)$	$U(n)$	$\frac{n(n-1)}{2}$	n
CI	$Sp(n,\mathbb{R})$	$U(n)$	$\frac{n(n+1)}{2}$	n
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{so}(10) \oplus \mathbb{R}$	16	6
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_6 \oplus \mathbb{R}$	27	7

For D in Table 1 to be a Reinhardt domain, we clearly have as a necessary condition the inequality:

$$\dim(T) \geq \dim_{\mathbb{C}}(D). \tag{8.1}$$

Let us now consider the cases where this might occur in Table 1.

AIII. The condition (8.1) holds if and only if $\min(p,q) = 1$, which clearly corresponds to the unit ball of dimension $\max(p,q)$.

BDI(2,q). In this case the condition (8.1) holds if and only if $q = 1$. This corresponds to the bounded symmetric domain whose group of biholomorphisms is, up to a finite covering, $SO_0(2,1)$. Since the Lie algebras $\mathfrak{so}(2,1)$ and $\mathfrak{su}(1,1)$ are isomorphic, the bounded symmetric domain of type **BDI(2,1)** is the unit disc in the complex plane.

DIII. In this case the condition (8.1) holds only for $n = 2$ or 3. The Lie algebras of the corresponding groups G_0 are $\mathfrak{so}^*(4)$ and $\mathfrak{so}^*(6)$. There are well known isomorphisms $\mathfrak{so}^*(4) \cong \mathfrak{su}(2) \times \mathfrak{su}(1,1)$ and $\mathfrak{so}^*(6) \cong \mathfrak{su}(3,1)$ (see [5]). We also recall that $\mathfrak{u}(n) \cong \mathfrak{su}(n) \oplus \mathbb{R}$, for every n. Hence, we conclude that type **DIII** for $n = 2$ and 3 defines the unit disc in the complex plane and the unit ball in $\mathbb{C}^3$, respectively.

CI. In this case the condition (8.1) holds only for $n = 1$, which yields the unit disk with an argument as above using the fact that $\mathfrak{sp}(1,\mathbb{R})$ is isomorphic to $\mathfrak{su}(1,1)$ (see [5]).

EIII, EVII. A simple inspection shows that in these cases the condition (8.1) cannot hold.

This completes the proof of Theorem 8.1. □

Now the results of the previous sections lead directly to the following statements:

1. On the subset $\widehat{\mathbb{B}}^n$ the $\mathbb{T}^n$-action defines a Lagrangian foliation $\mathcal{O}$.
2. The orthogonal complement $T\mathcal{O}^\perp$ is integrable in $\widehat{\mathbb{B}}^n$ to a foliation totally geodesic Lagrangian foliation $\mathcal{P}$.
3. The pair of foliations $\mathcal{O}$ and $\mathcal{P}$ define the polar coordinates in $\widehat{\mathbb{B}}^n$, which in turn yields the commutative C^*- algebra of Toeplitz operators whose symbols are constant on the leaves of $\mathcal{O}$.

In what follows we will normalize the (Hermitian) Bergman metric on the unit ball to the following expression:

$$ds^2 = \frac{4}{1 - \sum_{k=1}^n |z_k|^2} \left(\sum_{k=1}^n dz^k \otimes d\overline{z}^k + \sum_{k,l=1}^n \frac{\overline{z}_k z_l \, dz^k \otimes d\overline{z}^l}{1 - \sum_{k=1}^n |z_k|^2} \right).$$

which differs from the usual Bergman metric as considered in the proof of Theorem 7.2 by a factor of $(n+1)/4$. The advantage of this normalization is that the sectional curvature varies in the interval $[-1, -1/4]$, while with the metric as defined in the proof of Theorem 7.2 the sectional curvature varies in the interval $[-4/(n+1), -1/(n+1)]$.

We will now compute some values of the second fundamental form for the foliation $\mathcal{O}$ of the unit ball. First, we recall the notion of complex geodesic and some of its properties.

Definition 8.2. A complex geodesic in $\mathbb{B}^n$ is a biholomorphic map $\varphi : \mathbb{D} \to \mathbb{D}'$ where $\mathbb{D}$ is the unit disc and $\mathbb{D}' = \mathbb{B}^n \cap L$ for some complex affine line L in $\mathbb{C}^n$.

It is well known that complex geodesics are always totally geodesic maps. Furthermore, the images of complex geodesics are precisely the closed totally geodesic complex submanifolds of (complex) dimension 1 in $\mathbb{B}^n$ (see [2]).

The next result shows that some of the orbits of the $\mathbb{T}^n$-action on the unit ball integrate the vector fields of the framing $\left(\frac{\partial}{\partial\theta_k}\right)_{k=1}^n$ from Corollary 7.9. Its proof is a straightforward computation.

Lemma 8.3. *For every $k, 1 = 1, \ldots, n$ with $k \neq l$, the curves:*

$$\begin{aligned} \gamma_{z,k}(s) &= (z_1, \ldots, z_{k-1}, e^{is} z_k, z_{k+1}, \ldots, z_n) \\ \gamma_{z,kl}(s) &= (z_1, \ldots, z_{k-1}, e^{is} z_k, z_{k+1}, \ldots, z_{l-1}, e^{is} z_l, z_{l+1}, \ldots, z_n) \end{aligned}$$

are integral curves of the vector fields $\frac{\partial}{\partial\theta_k}$ and $\frac{\partial}{\partial\theta_k} + \frac{\partial}{\partial\theta_l}$, respectively.

Proof. By the definition of polar coordinates it is clear that the flows that integrate $\frac{\partial}{\partial\theta_k}$ and $\frac{\partial}{\partial\theta_k} + \frac{\partial}{\partial\theta_l}$ are given by:

$$\begin{aligned} z &\mapsto (z_1, \ldots, z_{k-1}, e^{is} z_k, z_{k+1}, \ldots, z_n), \\ z &\mapsto (z_1, \ldots, z_{k-1}, e^{is} z_k, z_{k+1}, \ldots, z_{l-1}, e^{is} z_l, z_{l+1}, \ldots, z_n) \end{aligned}$$

from which the conclusion is clear. □

Let us define the following vector fields on $\widehat{\mathbb{B}}^n$:

$$\begin{aligned} Q_k &= \mathrm{II}\left(\frac{\partial}{\partial\theta_k}, \frac{\partial}{\partial\theta_k}\right) \\ Q_{kl} &= \mathrm{II}\left(\frac{\partial}{\partial\theta_k} + \frac{\partial}{\partial\theta_l}, \frac{\partial}{\partial\theta_k} + \frac{\partial}{\partial\theta_l}\right), \end{aligned}$$

then, by Proposition 5.4 and Corollary 7.9, such vector fields completely determine the second fundamental form II. We will compute Q_k and Q_{kl} using the curves defined in Lemma 8.3. To achieve this, for every $z \in \widehat{\mathbb{B}}^n$ we define the following complex geodesics:

$$\begin{aligned} \phi_{z,k}(w) &= (z_1, \ldots, z_{k-1}, R_k w, z_{k+1}, \ldots, z_n) \\ \phi_{z,kl}(w) &= \left(z_1, \ldots, z_{k-1}, R_{kl} w, z_{k+1}, \ldots, z_{l-1}, \frac{R_{kl} z_l}{z_k} w, z_{l+1}, \ldots, z_n\right) \end{aligned}$$

where $k, l = 1, \ldots, n$ with $k \neq l$ and:

$$\begin{aligned} R_k &= \sqrt{1 - \sum_{j \neq k} |z_j|^2} \\ R_{kl} &= \frac{|z_k| \sqrt{1 - \sum_{j \neq k,l} |z_j|^2}}{\sqrt{|z_k|^2 + |z_l|^2}}. \end{aligned}$$

Then we have the following easy to prove result.

Lemma 8.4. *For every $z \in \widehat{\mathbb{B}}^n$ the complex geodesics $\phi_{z,k}$, $\phi_{z,kl}$ satisfy:*

1. $\phi_{z,k}(z_k/R_k) = \phi_{z,kl}(z_k/R_{kl}) = z$ *for every* $k, l = 1, \ldots, n$ *with* $k \neq l$,
2. $\gamma_{z,k}(\mathbb{R}) \subset \phi_{z,k}(\mathbb{D})$ *and* $\gamma_{z,kl}(\mathbb{R}) \subset \phi_{z,kl}(\mathbb{D})$,

in other words, they pass through z and contain the curves from Lemma 8.3 *with the same indices.*

We now use the above to compute the value of the vector fields Q_k and Q_{kl}.

Lemma 8.5. *For every $z \in \widehat{\mathbb{B}}^n$ and $k, l = 1, \ldots, n$ with $k \neq l$ we have the following relations:*

1. $Q_k(z) = \gamma''_{z,k}(0)$ *and* $Q_{kl}(z) = \gamma''_{z,kl}(0)$, *where the acceleration is computed for the complex hyperbolic geometry of* $\mathbb{B}^n$,
2. $\gamma''_{z,k}(s) \in \mathbb{R}i\gamma'_{z,k}(s)$ *and* $\gamma''_{z,kl}(s) \in \mathbb{R}i\gamma'_{z,kl}(s)$ *for every* $s \in \mathbb{R}$; *in particular:*
$$Q_k(z) \in \mathbb{R}\frac{\partial}{\partial r_k}\Big|_z, \quad Q_{kl}(z) \in \mathbb{R}\left(\frac{\partial}{\partial r_k}\Big|_z + \frac{\partial}{\partial r_l}\Big|_z\right),$$
3. *the norms of Q_k and Q_{kl} are given by:*
$$\|Q_k(z)\| = C_k(z)\|\gamma'_{z,k}(0)\|^2, \quad \|Q_{kl}(z)\| = C_{kl}(z)\|\gamma'_{z,kl}(0)\|^2,$$
where $C_k(z)$ and $C_{kl}(z)$ are the geodesic curvatures of $\gamma_{z,k}$ and $\gamma_{z,kl}$, respectively, considered as curves in the images of the complex geodesics $\phi_{z,k}$ and $\phi_{z,kl}$, respectively, endowed with the metric inherited from $\mathbb{B}^n$.

Proof. First we observe that in $\widehat{\mathbb{B}}^n$ the leaves of the foliation $\mathcal{O}$ by $\mathbb{T}^n$-orbits are diffeomorphic to $\mathbb{T}^n$ under the action map. In particular, with respect to such diffeomorphisms, the metric of $\mathbb{B}^n$ restricted to any such $\mathbb{T}^n$-orbit is left invariant. For such metrics on abelian Lie groups it is well known that the geodesics are precisely the one parameter subgroups and their translations (see [5]). Since the curves $\gamma_{z,k}$, $\gamma_{z,kl}$ correspond to one parameter groups in $\mathbb{T}^n$ it follows that they define geodesics in the leaf of $\mathcal{O}$ through z. Then, by well known results on the geometry of Riemannian submanifolds (see [9]) it follows that the accelerations $\gamma''_{z,k}$, $\gamma''_{z,kl}$ as computed in $\mathbb{B}^n$ are everywhere perpendicular to the leaves of $\mathcal{O}$, in other words they are everywhere horizontal.

By the remarks in Section 5 and the definition of Q_k and Q_{kl} we have:
$$\begin{aligned} Q_k(z) &= \mathcal{H}(\widehat{\nabla}_{\gamma'_{z,k}(0)}\gamma'_{z,k}) = \mathcal{H}(\gamma''_{z,k}(0)) = \gamma''_{z,k}(0) \\ Q_{kl}(z) &= \mathcal{H}(\widehat{\nabla}_{\gamma'_{z,kl}(0)}\gamma'_{z,kl}) = \mathcal{H}(\gamma''_{z,kl}(0)) = \gamma''_{z,kl}(0), \end{aligned}$$
for $\widehat{\nabla}$ the connection of $\mathbb{B}^n$, and where the last identities follow from the remarks in the previous paragraph. This proves (1).

Next observe that since the curves $\gamma_{z,k}$, $\gamma_{z,kl}$ are geodesics in some leaf of $\mathcal{O}$ it follows that they are up to a constant parameterized by arc-length. On the other hand, by Lemma 8.4 the curves $\gamma_{z,k}$, $\gamma_{z,kl}$ lie in complex geodesics which, as we observed before, define totally geodesic submanifolds of $\mathbb{B}^n$. In particular, their accelerations $\gamma''_{z,k}$, $\gamma''_{z,kl}$ as computed in $\mathbb{B}^n$ are the same as computed in the (images) of the complex geodesics that contain them. Complex geodesics are isometric to the unit disk, and for the latter any curve γ which is parameterized up to a constant by arc length satisfies $\gamma''(s) \in \mathbb{R}i\gamma'(s)$. This implies the first part of (2). For the second part of (2) it is enough to note that:

$$\begin{aligned} i\gamma'_{z,k}(s) &= i\frac{\partial}{\partial \theta_k}\Big|_{\gamma_{z,k}(s)} \in \mathbb{R}\frac{\partial}{\partial r_k}\Big|_{\gamma_{z,k}(s)} \\ i\gamma'_{z,kl}(s) &= i\left(\frac{\partial}{\partial \theta_k}\Big|_{\gamma_{z,kl}(s)} + \frac{\partial}{\partial \theta_l}\Big|_{\gamma_{z,kl}(s)}\right) \in \mathbb{R}\left(\frac{\partial}{\partial r_k}\Big|_{\gamma_{z,kl}(s)} + \frac{\partial}{\partial r_l}\Big|_{\gamma_{z,kl}(s)}\right), \end{aligned}$$

and so we obtain (2) by applying (1).

By the definition of the geodesic curvature (see [4] and its references) we have:

$$\begin{aligned} C_k(z) &= \frac{\|\gamma''_{z,k}(0)\|}{\|\gamma'_{z,k}(0)\|^2} \\ C_{kl}(z) &= \frac{\|\gamma''_{z,kl}(0)\|}{\|\gamma'_{z,kl}(0)\|^2}, \end{aligned}$$

where we have used the fact that the norm of vectors and the acceleration of curves in a (image of a) complex geodesic in $\mathbb{B}^n$ computed in $\mathbb{B}^n$ or the complex geodesic yield the same result. Given the above identities, (3) follows from (1). $\square$

The next result computes specific values for the second fundamental form of the foliation $\mathcal{O}$ for the unit ball. By Proposition 5.4 such values completely determine the second fundamental form. We observe that in the first two parts of the statement we obtain a very explicit expression for the second fundamental form of the foliation $\mathcal{O}$. Note that by part (3) of Lemma 8.5, the geodesic curvatures C_k and C_{kl} correspond to the norms of the values of Q_k and Q_{kl}, respectively, normalized so that they only depend on the direction of $\frac{\partial}{\partial \theta_k}$ and $\frac{\partial}{\partial \theta_k} + \frac{\partial}{\partial \theta_l}$, respectively. In view of this, the last part of the statement allows us to understand the asymptotic behavior of the curvature of the leaves of $\mathcal{O}$ as they move towards the origin or the boundary of $\widehat{\mathbb{B}}^n$. We observe as well that this result generalizes our geometric description of the elliptic model case in the unit disk found in [4].

Theorem 8.6. *For every $z \in \widehat{\mathbb{B}}^n$, let $r = (r_1, \dots, r_n) = (|z_1|, \dots, |z_n|)$, and consider the curves $\gamma_{z,k}$, $\gamma_{z,kl}$ and the complex geodesics $\phi_{z,k}$, $\phi_{z,kl}$ defined above. Then:*

1. *The vector fields Q_k and Q_{kl} are given by:*

$$\begin{aligned} Q_k(z) &= -C_k(z)\left\|\frac{\partial}{\partial \theta_k}\Big|_z\right\|^2 \left\|\frac{\partial}{\partial r_k}\Big|_z\right\|^{-1}\left(\frac{\partial}{\partial r_k}\Big|_z\right) \\ Q_{kl}(z) &= -C_{kl}(z)\left\|\frac{\partial}{\partial \theta_k}\Big|_z + \frac{\partial}{\partial \theta_l}\Big|_z\right\|^2 \left\|\frac{\partial}{\partial r_k}\Big|_z + \frac{\partial}{\partial r_l}\Big|_z\right\|^{-1}\left(\frac{\partial}{\partial r_k}\Big|_z + \frac{\partial}{\partial r_l}\Big|_z\right). \end{aligned}$$

2. *The geodesic curvatures* $C_k(z)$ *and* $C_{kl}(z)$ *at* z *defined in Lemma* 8.5 *are given by:*

$$\begin{aligned} C_k(z) &= \frac{r_k^2 + \left(1 - \sum_{j\neq k} r_j^2\right)}{2r_k\sqrt{1 - \sum_{j\neq k} r_j^2}} \\ C_{kl}(z) &= \frac{r_k^2 + r_l^2 + \left(1 - \sum_{j\neq k,l} r_j^2\right)}{2\sqrt{r_k^2 + r_l^2}\sqrt{1 - \sum_{j\neq k,l} r_j^2}}, \end{aligned}$$

in particular, such geodesic curvatures lie in the interval $(1, +\infty)$ *and achieve all values therein.*

3. *The geodesic curvatures* $C_k(z)$ *and* $C_{kl}(z)$ *have the following asymptotic behavior:*

$$\begin{aligned} C_k(z), C_{kl}(z) &\to +\infty, \quad \text{as } |z| \to 0, \\ C_k(z) &\to 1, \quad \text{as } z \to u, \\ C_{kl}(z) &\to 1, \quad \text{as } z \to v, \end{aligned}$$

for any $u, v \in \partial\mathbb{B}^n$ *such that* $u_k \neq 0$ *and* $|v_k|^2 + |v_l|^2 \neq 0$*, respectively.*

Proof. Up to a sign, (1) essentially follows from (2) and (3) in Lemma 8.5. The negative sign comes from the fact that, in the proof of Lemma 8.5, the accelerations of $\gamma_{z,k}$, $\gamma_{z,kl}$ point towards the origin in the complex geodesics that contain them and the vector fields

$$\frac{\partial}{\partial r_k}, \quad \frac{\partial}{\partial r_k} + \frac{\partial}{\partial r_l}$$

point away from the origin.

To prove (2), let $\phi_{z,k}$, $\phi_{z,kl}$ be the complex geodesics considered before. Then the inverse images of the curves $\gamma_{z,k}$, $\gamma_{z,kl}$ with respect to such maps are easily seen to be circles in $\mathbb{D}$ centered at the origin with Euclidean radius:

$$s_k = \frac{r_k}{R_k} = \frac{r_k}{\sqrt{1 - \sum_{j\neq k} r_j^2}} \quad \text{and} \quad s_{kl} = \frac{r_k}{R_{kl}} = \frac{\sqrt{r_k^2 + r_l^2}}{\sqrt{1 - \sum_{j\neq k,l} r_j^2}},$$

respectively. Next, we observe that the geodesic curvature $C(s)$ of the circle with Euclidean radius s in the unit disk $\mathbb{D}$ with the metric:

$$\frac{4(dx^2 + dy^2)}{(1 - (x^2 + y^2))^2}$$

is given by the formula:

$$C(s) = \frac{1 + s^2}{2s}.$$

This follows from two facts found in [2]. The hyperbolic radius ρ of a circle centered at the origin satisfies $\cosh^2(\rho/2) = 1/(1 - s^2)$, where s is the Euclidean radius. And the geodesic curvature of such a circle is given by $\coth(\rho)$. The first can be deduce from the expression for the hyperbolic distance found in subsection 1.4.1 of [2] and the second is stated in subsection 1.4.2 of the same reference.

Given the above formula for $C(s)$ a simple substitution provides the required expressions for C_k and C_{kl}. Finally, (3) is a consequence of these expressions. $\square$

We recall that the velocity and acceleration of curves in a manifold do not change when we renormalize the metric of the manifold by a constant multiple. More generally, the second fundamental form of a submanifold does not change either by such renormalizations (see [9]). However, the geodesic curvatures as defined above involve the metric and so they are rescaled when we renormalize the metric by a constant.

In particular, for the Bergman metric on $\mathbb{B}^n$ (i.e. without normalizing to have sectional curvature in the interval $[-1, -1/4]$) which is given by

$$ds^2_{\mathbb{B}^n} = \frac{n+1}{1 - \sum_{k=1}^n |z_k|^2} \left(\sum_{k=1}^n dz^k \otimes d\overline{z}^k + \sum_{k,l=1}^n \frac{\overline{z}_k z_l \, dz^k \otimes d\overline{z}^l}{1 - \sum_{k=1}^n |z_k|^2} \right).$$

the tangent vectors $\gamma'_{z,k}(0)$, $\gamma''_{z,k}(0)$ and $Q_k(z) = \mathrm{II}\left(\frac{\partial}{\partial\theta_k}\Big|_z, \frac{\partial}{\partial\theta_k}\Big|_z \right)$ as defined above have the same values, but computing the geodesic curvatures of $\gamma_{z,k}$ involve applying a renormalized metric and the corresponding values are rescaled. In the next result we write down the geodesic curvatures of the curves $\gamma_{z,k}$ for the Bergman metric and describe its asymptotic behavior. We also express such curvatures in terms of the second fundamental form II. These facts will allow us to compare our present situation with a more general asymptotic behavior discussed in the next section.

Theorem 8.7. *Let $h_{\mathbb{B}^n}$ be the Riemannian metric associated to the Bergman metric of $\mathbb{B}^n$ given as above and denote with $\|\cdot\|_{\mathbb{B}^n}$ the norm that it defines on tangent vectors. Then, for every $z \in \widehat{\mathbb{B}}^n$ the geodesic curvature $\widehat{C}_k(z)$ of the curve $\gamma_{z,k}$ at z for the metric $h_{\mathbb{B}^n}$ satisfies the relations:*

1. $\widehat{C}_k(z) = -\left\| \frac{\partial}{\partial\theta_k}\Big|_z \right\|_{\mathbb{B}^n}^{-2} \left\| \frac{\partial}{\partial r_k}\Big|_z \right\|_{\mathbb{B}^n}^{-1} h_{\mathbb{B}^n}\left(Q_k(z), \frac{\partial}{\partial r_k}\Big|_z \right)$,
2. $\widehat{C}_k(z) = \frac{2}{\sqrt{n+1}} C_k(z)$, *where C_k is given as in Theorem 8.6.*

In particular, $\widehat{C}_k(z)$ is, up to a sign, the norm of the orthogonal projection of the vector $\left\| \frac{\partial}{\partial\theta_k}\Big|_z \right\|_{\mathbb{B}^n}^{-2} Q_k(z)$ onto $\frac{\partial}{\partial r_k}\Big|_z$ with respect to $h_{\mathbb{B}^n}$. And we also have:

$$\widehat{C}_k(z) \to \frac{2}{\sqrt{n+1}}, \qquad \textit{as } z \to u.$$

for any $u \in \partial\mathbb{B}^n$ such that $u_k \neq 0$.

Proof. The first relation follows from the definition of the geodesic curvature as above applied to the new metric $h_{\mathbb{B}^n}$, the fact that:

$$Q_k(z) = \mathrm{II}\left(\frac{\partial}{\partial\theta_k}\Big|_z, \frac{\partial}{\partial\theta_k}\Big|_z \right)$$

and the corresponding relation of (1) in Theorem 8.6 for $\widehat{C}_k$.

The second relation is a consequence of the fact that $h_{\mathbb{B}^n} = \frac{n+1}{4} h$ for h the Riemannian metric on $\mathbb{B}^n$ rescaled so that its sectional curvature lies in $[-1, -1/4]$. □

We specify now the results obtained in Sections 2 and 3 for the unit ball $\mathbb{B}^n$. The base $\tau(\mathbb{B}^n)$ of $\mathbb{B}^n$ has obviously the form

$$\tau(\mathbb{B}^n) = \{r = (r_1, \dots, r_n) : r^2 = r_1^2 + \cdots + r_n^2 \in [0,1)\}.$$

As a custom in operator theory (see, for example, [15]), introduce the family of weights

$$\mu_\lambda(|z|) = c_\lambda \, (1 - |z|^2)^\lambda,$$

where the normalizing constant

$$c_\lambda = \frac{\Gamma(n + \lambda + 1)}{\pi^n \Gamma(\lambda + 1)}$$

is chosen so that $\mu_\lambda(|z|)dv(z)$ is a probability measure in $\mathbb{B}^n$.

Introduce $L_2(\mathbb{B}^n, \mu_\lambda)$ and its Bergman subspace $\mathcal{A}^2_\lambda(\mathbb{B}^n) = \mathcal{A}^2_{\mu_\lambda}(\mathbb{B}^n)$. It is well known (see, for example, [15]), that the Bergman projection B_λ of $L_2(\mathbb{B}^n, \mu_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ has the form

$$(B_\lambda \varphi)(z) = \int_{\mathbb{B}^n} \varphi(\zeta) \, K_\lambda(z, \zeta) \, \mu_\lambda(|\zeta|) dv(\zeta),$$

where the (weighted) Bergman kernel is given by

$$K_\lambda(z, \zeta) = \frac{1}{(1 - \sum_{k=1}^n z_k \overline{\zeta}_k)^{n+1+\lambda}}.$$

To calculate the constant α_p, $p = (p_1, \dots, p_n) \in \mathbb{Z}^n_+$, see (2.1), consider the integral

$$\begin{aligned} \int_{\mathbb{B}^n} |z^p|^2 \mu_\lambda(|z|) dv(z) &= \int_{\mathbb{B}^n} |z_1|^{2p_1} \cdot \dots \cdot |z_n|^{2p_n} \mu_\lambda(r) dv(z) \\ &= \int_{\mathbb{T}^n} \prod_{k=1}^n \frac{dt_k}{it_k} \int_{\tau(\mathbb{B}^n)} r_1^{2p_1} \cdot \dots \cdot r_n^{2p_n} \mu_\lambda(r) \prod_{k=1}^n r_k dr_k \\ &= (2\pi)^n \alpha_p^{-2}. \end{aligned}$$

From the other hand side, by [15], Lemma 1.11, we have

$$\int_{\mathbb{B}^n} |z^p|^2 \mu_\lambda(|z|) dv(z) = \frac{p! \, \Gamma(n + \lambda + 1)}{\Gamma(n + |p| + \lambda + 1)},$$

that is,

$$\alpha_p = \left(\frac{(2\pi)^n \, \Gamma(n + |p| + \lambda + 1)}{p! \, \Gamma(n + \lambda + 1)} \right)^{1/2}.$$

Now Theorem 3.1 for the case of the unit ball reads as follows.

Theorem 8.8. *Let $a = a(r)$ be a bounded measurable separately radial function. Then the Toeplitz operator T_a acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is unitary equivalent to the multiplication operator $\gamma_a I = R T_a R^*$ acting on $l_2(\mathbb{Z}^n_+)$, where R and R^* are given by (2.3) and (2.2) respectively. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(p)\}_{p\in\mathbb{Z}^n_+}$ is given by*

$$\begin{aligned}\gamma_{a,\lambda}(p) &= \frac{2^n\,\Gamma(n+|p|+\lambda+1)}{p!\,\Gamma(\lambda+1)}\int_{\tau(\mathbb{B}^n)} a(r)\,r^{2p}\,(1-r^2)^\lambda \prod_{k=1}^n r_k dr_k \\ &= \frac{\Gamma(n+|p|+\lambda+1)}{p!\,\Gamma(\lambda+1)}\int_{\Delta(\mathbb{B}^n)} a(\sqrt{r})\,r^p\,(1-(r_1+\cdots+r_n))^\lambda\,dr, \quad p\in\mathbb{Z}^n_+,\end{aligned}$$

where $\Delta(\mathbb{B}^n) = \{r = (r_1,\ldots,r_n) : r_1+\cdots+r_n \in [0,1),\ r_k \geq 0,\ k=1,\ldots,n\}$, $dr = dr_1\cdots dr_n$, and $\sqrt{r} = (\sqrt{r_1},\ldots,\sqrt{r_n})$.

9. Asymptotic geometric behavior of the $\mathbb{T}^n$-orbits in Reinhardt domains

As before, let D be a bounded logarithmically convex complete Reinhardt domain with Bergman metric ds^2_D, associated Riemannian metric h_D and with $\|\cdot\|_D$ denoting the norm defined by h_D on tangent vectors.

Also, we will continue denoting with II the second fundamental form of the foliation $\mathcal{O}$ by $\mathbb{T}^n$-orbits in $\widehat{D}$. As in the case of the unit ball, by Proposition 5.4, II is completely determined by the vector fields:

$$\begin{aligned}Q_k &= \mathrm{II}\left(\frac{\partial}{\partial\theta_k},\frac{\partial}{\partial\theta_k}\right)\\ Q_{kl} &= \mathrm{II}\left(\frac{\partial}{\partial\theta_k}+\frac{\partial}{\partial\theta_l},\frac{\partial}{\partial\theta_k}+\frac{\partial}{\partial\theta_l}\right).\end{aligned}$$

The norm of such vector fields was computed in the previous section for the unit ball and such norm was related to the geodesic curvature of suitable circles contained in complex geodesics. In this section we will study the asymptotic behavior towards the boundary of similar values for a more general Reinhardt domain.

As in the case of the unit ball, on our given Reinhardt domain, we will consider for every $z\in\widehat{D}$ and $k=1,\ldots,n$ the curve:

$$\gamma_{z,k}(s) = (z_1,\ldots,z_{k-1},e^{is}z_k,z_{k+1},\ldots,z_n).$$

Then, the proofs of Lemmas 8.3 and 8.5 apply to our current more general setup without change to conclude that $\gamma_{z,k}$ is an integral curve of $\frac{\partial}{\partial\theta_k}$ and that we can write:

$$Q_k(z) = \mathrm{II}\,(\gamma'_{z,k}(0),\gamma'_{z,k}(0)) = \gamma''_{z,k}(0).$$

As in the case of the unit ball, to better understand the asymptotic behavior of the values of Q_k one considers its normalized value obtained by dividing by

$\|\gamma'_{z,k}(0)\|_D^2 = \|\frac{\partial}{\partial\theta_k}|_z\|_D^2$, i.e.:

$$\left\|\frac{\partial}{\partial\theta_k}\Big|_z\right\|_D^{-2} Q_k(z) = \left\|\frac{\partial}{\partial\theta_k}\Big|_z\right\|_D^{-2} \mathrm{II}\left(\frac{\partial}{\partial\theta_k}\Big|_z, \frac{\partial}{\partial\theta_k}\Big|_z\right) = \frac{\gamma''_{z,k}(0)}{\|\gamma'_{z,k}(0)\|_D^2},$$

which now depends only on the direction associated to $\gamma'_{z,k}(0) = \frac{\partial}{\partial\theta_k}|_z$ and not on its magnitude. Moreover, the above identities show that $\left\|\frac{\partial}{\partial\theta_k}\Big|_z\right\|_D^{-2} Q_k(z)$ measures both the extrinsic curvature of the foliation $\mathcal{O}$, given by II, and the curvature of $\gamma_{z,k}$, given by its acceleration. For the unit ball it was proved that such vector field is collinear with $\frac{\partial}{\partial r_k}$, and so to measure its magnitude in that case it was enough to consider the norm of its orthogonal projection onto $\frac{\partial}{\partial r_k}$. In our more general setup, $\left\|\frac{\partial}{\partial\theta_k}\right\|_D^{-2} Q_k$ may not be collinear with $\frac{\partial}{\partial r_k}$, but we can still consider the properties of the orthogonal projection of the first onto the latter.

The previous discussion suggests to define for every $z \in \widehat{D}$ and $k = 1, \ldots, n$:

$$\widehat{C}_k(z) = -\left\|\frac{\partial}{\partial\theta_k}\Big|_z\right\|_D^{-2} \left\|\frac{\partial}{\partial r_k}\Big|_z\right\|_D^{-1} h_D\left(Q_k(z), \frac{\partial}{\partial r_k}\Big|_z\right),$$

which thus provides a measure of both the extrinsic curvature of the foliation $\mathcal{O}$ on $\widehat{D}$ and the curvature of $\gamma_{z,k}$.

Note that for the unit ball endowed with the Bergman metric, Theorem 8.7 shows that $\widehat{C}_k(z)$ is precisely the geodesic curvature of $\gamma_{z,k}$ in the complex geodesic $\phi_{z,k}$ considered in the previous section. Moreover, such Theorem 8.7 describes the asymptotic behavior of $\widehat{C}_k$ towards the boundary in the case of the unit ball. The main goal of this section is to prove that such asymptotic behavior remains valid for suitable domains. More precisely, we have the following result. We recall that D is said to have δ as a defining function if $D = \{z \in \mathbb{C}^n : \delta(z) < 0\}$.

Theorem 9.1. *Let D be a bounded strictly pseudoconvex complete Reinhardt domain with smooth boundary and with a smooth defining function δ. Then:*

$$\widehat{C}_k(z) \to \frac{2}{\sqrt{n+1}}, \qquad \text{as } z \to u,$$

for any $u \in \partial D$ such that $u_k \neq 0$ and $\frac{\partial\delta}{\partial r_k}(u) \neq 0$.

We observe that for the unit ball we can take $\delta(z) = -1 + \sum_{j=1}^n |z_j|^2 = -1 + \sum_{j=1}^n r_j^2$, and so the conditions $u_k \neq 0$ and $\frac{\partial\delta}{\partial r_k}(u) \neq 0$ are equivalent in this case.

Note that Theorems 9.1 and 8.7 together show that, under suitable convexity and smoothness conditions on the domain D, the extrinsic geometry of the foliation $\mathcal{O}$ in $\widehat{D}$ has exactly the same asymptotic behavior towards the boundary as the one found for the unit ball, at least with respect to the values of Q_k.

To prove Theorem 9.1 we will use the expression of the metric h_D in terms of the Bergman kernel K_D from Theorem 7.2 and the following celebrated result by

C. Fefferman that describes the Bergman kernel of strictly pseudoconvex domains with smooth boundary. This result appears as a Corollary in page 45 of [1].

Theorem 9.2 (C. Fefferman). *If D is a strictly pseudoconvex domain with smooth boundary and smooth defining function δ, then there exists $\varphi, \psi \in C^\infty(\overline{D})$ with φ nonvanishing in ∂D, such that:*

$$K_D(z,z) = \varphi(z)(-\delta(z))^{-(n+1)} + \psi(z)\log(-\delta(z))$$

for every $z \in D$.

We first express the value of $\widehat{C}_k$ in terms of the Bergman kernel K_D.

Lemma 9.3. *Let D be a bounded logarithmically convex complete Reinhardt domain with Bergman kernel K_D. Then:*

$$\widehat{C}_k = \frac{C'}{C''},$$

where

$$\begin{aligned} C' &= 2\left(\frac{\partial K_D}{\partial r_k}\right)^3 - 3K_D\frac{\partial K_D}{\partial r_k}\frac{\partial^2 K_D}{\partial r_k^2} + K_D^2\frac{\partial^3 K_D}{\partial r_k^3} - \frac{3K_D}{r_k}\left(\frac{\partial K_D}{\partial r_k}\right)^2 \\ &+ \frac{3K_D^2}{r_k}\frac{\partial^2 K_D}{\partial r_k^2} + \frac{K_D^2}{r_k^2}\frac{\partial K_D}{\partial r_k}, \\ C'' &= \left(-\left(\frac{\partial K_D}{\partial r_k}\right)^2 + K_D\frac{\partial^2 K_D}{\partial r_k^2} + \frac{K_D}{r_k}\frac{\partial K_D}{\partial r_k}\right)^{\frac{3}{2}}, \end{aligned}$$

and where K_D and its partial derivatives are computed for the function $z \mapsto K_D(z,z)$.

Proof. Let us denote with $\Gamma^{r_j}_{\theta_k\theta_l}, \dots,$ the Schwarz-Christoffel symbols for the Levi-Civita connection of the Riemannian manifold (D, h_D) and with $h_{\theta_k\theta_l}, \dots,$ the coordinate functions of the metric. By Corollary 7.9 and the definition of II it follows that:

$$Q_k(z) = \sum_{l=1}^n \Gamma^{r_l}_{\theta_k\theta_k}\frac{\partial}{\partial r_l}.$$

By using the well known formula that expresses the Schwarz-Christoffel symbols in terms of the functions $h_{\theta_k\theta_l}, \dots,$ and its partial derivatives (see [9]) we have:

$$\Gamma^{r_l}_{\theta_k\theta_k} = -\frac{1}{2}\sum_{j=1}^n h^{r_l r_j}\frac{\partial h_{\theta_k\theta_k}}{\partial r_j},$$

where, as usual, $h^{r_l r_j}$ denotes the entries of the inverse of the matrix $(h_{r_l r_j})_{lj}$. We have used here that, by Theorem 7.2, the functions $h_{\theta_k r_l} = 0$. Hence, it follows that:

$$\begin{aligned}
\widehat{C}_k &= -\left\|\frac{\partial}{\partial\theta_k}\Big|_z\right\|_D^{-2}\left\|\frac{\partial}{\partial r_k}\Big|_z\right\|_D^{-1} h_D\left(Q_k(z), \frac{\partial}{\partial r_k}\Big|_z\right) \\
&= \frac{1}{2}h_{\theta_k\theta_k}^{-1}h_{r_kr_k}^{-1/2}\sum_{l,j=1}^{n} h_{r_kr_l}h^{r_lr_j}\frac{\partial h_{\theta_k\theta_k}}{\partial r_j} \\
&= \frac{1}{2}h_{\theta_k\theta_k}^{-1}h_{r_kr_k}^{-1/2}\frac{\partial h_{\theta_k\theta_k}}{\partial r_k}.
\end{aligned}$$

Again by Theorem 7.2 we have $h_{r_kr_k} = F_{kk}$ and $h_{\theta_k\theta_k} = r_k^2F_{kk}$, where:

$$F_{kk}(z) = \frac{1}{4}\left(\frac{\partial^2}{\partial r_k^2} + \frac{1}{r_k}\frac{\partial}{\partial r_k}\right)\log K_D(z,z), \tag{9.1}$$

from which we obtain:

$$\widehat{C}_k = \frac{1}{2F_{kk}^{3/2}}\left(\frac{\partial F_{kk}}{\partial r_k} + \frac{2}{r_k}F_{kk}\right). \tag{9.2}$$

Then the result follows by computing F_{kk} and $\frac{\partial F_{kk}}{\partial r_k}$ in terms of K_D with the use of equation (9.1) and replacing into equation (9.2). □

The following result can be proved easily using induction.

Lemma 9.4. *Let D be a strictly pseudoconvex domain with smooth boundary and smooth defining function δ. Let $\varphi, \psi \in C^\infty(\overline{D})$ be the smooth functions from Theorem 9.2. Then, for every $k = 1, \dots, n$ and $j \geq 0$ we have:*

$$\frac{\partial^j K_D}{\partial r_k^j} = \sum_{l=0}^{j}\varphi_{jl}(-\delta)^{-(n+1+l)} + \sum_{l=1}^{j}\psi_{jl}\delta^{-l} + \psi_{j0}\log(-\delta),$$

where the partial derivatives are computed for the function $z \mapsto K_D(z,z)$, and the functions φ_{jl}, ψ_{jl} are given inductively by the following conditions:

1. $\varphi_{00} = \varphi$, $\psi_{00} = \psi$,
2. $\varphi_{jl} = \psi_{jl} = 0$ *if either j or l is negative,*
3. *for $j \geq 1$:*

$$\begin{aligned}
\varphi_{jj} &= (n+j)\varphi_{j-1,j-1}\frac{\partial\delta}{\partial r_k} \\
\psi_{jj} &= -(j-1)\psi_{j-1,j-1}\frac{\partial\delta}{\partial r_k},
\end{aligned}$$

4. *for $0 \leq l < j$:*

$$\begin{aligned}
\varphi_{jl} &= \frac{\partial\varphi_{j-1,l}}{\partial r_k} + (n+l)\varphi_{j-1,l-1}\frac{\partial\delta}{\partial r_k} \\
\psi_{jl} &= \frac{\partial\psi_{j-1,l}}{\partial r_k} - (l-1)\psi_{j-1,l-1}\frac{\partial\delta}{\partial r_k},
\end{aligned}$$

The next result provides an expression of $\widehat{C}_k$ in terms of the defining function δ. We observe that Theorem 9.1 is now an easy consequence of such expression.

Theorem 9.5. *Let D be a bounded strictly pseudoconvex complete Reinhardt domain with smooth boundary and with a smooth defining function δ. If φ is the function given by Theorem 9.2, then:*

$$\widehat{C}_k = \frac{a(-\delta)^{-(3n+6)} + b\delta^{-(3n+5)}}{(c(-\delta)^{-(2n+4)} + d\delta^{-(2n+3)})^{3/2}}$$

on $\widehat{D}$, where $a, b, c, d \in C^\infty(\widehat{D})$ satisfy:

1. $a = 2(n+1)\varphi^3 \left(\frac{\partial \delta}{\partial r_k}\right)^3$,
2. $c = (n+1)\varphi^2 \left(\frac{\partial \delta}{\partial r_k}\right)^2$,
3. *b, d extend continuously to $\{z \in \overline{D} : z_k \neq 0\}$.*

Proof. We use Lemmas 9.3 and 9.4 to express $\widehat{C}_k$ in terms of $\log(-\delta)$ and powers of δ with smooth coefficients in $\widehat{D}$. Then the result is simply a matter of identifying the coefficients in such expression. The functions a, c correspond to the lowest powers of δ in the numerator and the denominator, respectively. The functions b, d involve terms of the form δ^l and $\delta^l \log(-\delta)$ with $l \geq 1$ and powers of $1/r_k$, all of which can be extended continuously to $\{z \in \overline{D} : z_k \neq 0\}$. □

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