Commutative algebras of Toeplitz operators and Berezin quantization

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Abstract. It this survey paper we discuss the following two questions. The characterization of commutative C^*-algebras of Toeplitz operators acting on the weighted Bergman spaces over the unit disk, and the spectral properties of Toeplitz operators in dependence on the weight parameter.

1. Introduction

The paper is based on the talk given on the conference "Topics in Deformation Quantization and Noncommutative Structures" held at Cinvestav, Mexico City, September 2005, and is a survey of results from [**7, 8, 9, 10**], where further results, details, and proofs can be found.

Toeplitz operators with smooth (or continuous) symbols acting on weighted Bergman spaces over the unit disk, as well as C^*-algebras generated by such operators, naturally appear in the context of problems in mathematical physics. We mention here only: the quantum deformation of the algebra of continuous functions on the disk [**13**], and the Berezin quantization (in particular, on the hyperbolic plane); see, for example, [**2, 3, 4**].

Given a smooth symbol $a = a(z)$, the family of Toeplitz operators $T_a = \{T_a^{(h)}\}$, with $h \in (0, 1)$, is considered under the Berezin quantization procedure [**2, 3**]. For a fixed h the Toeplitz operator $T_a^{(h)}$ acts on the weighted Bergman space $\mathcal{A}_h^2(\mathbb{D})$, where h is the parameter characterizing the weight on $\mathcal{A}_h^2(\mathbb{D})$.

The same, as in a quantization procedure, weighted Bergman spaces appear naturally in many questions of complex analysis and operator theory. In the last cases a weight parameter is normally denoted by λ and runs through $(-1, +\infty)$. Note that the parameters connected by $\lambda + 2 = \frac{1}{h}$ define the same space.

The first question treated in the paper is the characterization of the commutative C^*-algebras of Toeplitz operators, acting on the weighted Bergman spaces. It was recently shown ([**19, 20**]) that apart from the known case of radial symbols in the classical (weightless) Bergman space $\mathcal{A}^2(\mathbb{D})$ there exists a rich family of

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commutative C^*-algebras of Toeplitz operators. Moreover, surprisingly it turns out that these commutative properties of Toeplitz operators do not depend at all on smoothness properties of the symbols: the corresponding symbols can be merely measurable. Furthermore ([**7, 8, 9**]), the above classes of symbols generate commutative C^*-algebras of Toeplitz operators on *each weighted Bergman space* $\mathcal{A}^2_h(\mathbb{D})$. The prime cause here appears to be the geometric configuration of level lines of symbols. All commutative C^*-algebras of Toeplitz operators discovered can be classified by pencils of geodesics on the unit disk, considered as the hyperbolic plane. More precise, given a pencil of geodesics, consider the set of symbols constant on the corresponding cycles, the orthogonal trajectories to geodesics forming the pencil. The C^*-algebra generated by Toeplitz operators with such symbols turns out to be commutative. Moreover [**10**], assuming some natural conditions on "richness" of symbol sets, the above mentioned classes of symbols are the only possible ones that generate commutative C^*-algebras of Toeplitz operators on each weighted Bergman space.

We note that the commutativity on each weighted Bergman space is of great importance and permits us to make use of the Berezin quantization procedure. At the same time to obtain the necessary information about potential symbols we need to calculate the second and third terms in the asymptotic expansion of a commutator. The first three terms of this expansion together provide us with exact geometric information: in order to generate a commutative C^*-algebra of Toeplitz operators on each weighted Bergman space the symbols must be constant on the cycles of some pencil of geodesics.

Another important question, which appear under the study of families of Toeplitz operators $T_a^{(h)}$ parameterized by h, is the study of their spectral properties in dependence on h, and especially the limit behavior of spectra under $h \to 0$. We treat this problem [**7, 8, 9**] using the (operator theory) weight parameter λ.

Although it seems to be quite impossible to get a reasonably complete answer to the mentioned question for general symbols, the essential progress can be achieved for the model cases of pencils and Toeplitz operators with symbols constant on the corresponding cycles. The key feature of such symbols, permitting us to get much more complete information that one obtained studying general symbols, is as follows. In each case of a commutative Toeplitz C^*-algebra the Toeplitz operators admit the spectral type representation, they are unitary equivalent to certain multiplication operators. This leads immediately to exact formulas for the spectrum for each value of λ. It turns out that for symbols constant on cycles, the spectra of Toeplitz operators tend to a certain limit set, when $\lambda \to \infty$, prescribed by the type of symbol considered, continuous, piecewise continuous, or oscillating.

We pass now to the exact definitions and statements.

2. Commutative algebras and pencils of geodesics

We use the standard Möbius invariant normalized measure on the unit disk $\mathbb{D}$

$$d\mu(z) = \frac{1}{\pi} \frac{dx\, dy}{(1-(x^2+y^2))^2}.$$

For $h \in (0,1)$, the weighted Bergman space $\mathcal{A}^2_h(\mathbb{D})$ (see, for example, [**4**]) is the space of analytic functions in $L_2(\mathbb{D}, d\mu_h)$, where

$$d\mu_h(z) = (\frac{1}{h} - 1)(1-|z|^2)^{\frac{1}{h}}\, d\mu(z),$$

and

$$\|f\|_h = \left(\int_{\mathbb{D}} |f(z)|^2 \, d\mu_h(z) \right)^{\frac{1}{2}}.$$

Alternative definition of the weighted Bergman spaces is common in operator theory (see, for example, [**12**]) and uses another parameter $\lambda \in (-1, \infty)$. The weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$ on the unit disk is the space of analytic functions in $L_2(\mathbb{D}, d\mu_\lambda)$, where

$$d\mu_\lambda = \frac{\lambda + 1}{\pi} (1 - |z|^2)^\lambda \, dv(z),$$

and $dv(x) = dxdy$ is the standard Lebesgue measure in $\mathbb{C}$.

For $\lambda + 2 = \frac{1}{h}$ we have the same spaces, and for $\lambda = 0$ or $h = \frac{1}{2}$ we have the classical weightless Bergman space (with normalized measure).

Recall that the orthogonal Bergman projection from $L_2(\mathbb{D}, d\mu_h)$ onto the Bergman space $\mathcal{A}^2_h(\mathbb{D})$ has the form (see, for example, [**4**]):

$$(B^{(h)}_{\mathbb{D}} f)(z) = \int_{\mathbb{D}} \frac{f(\zeta)}{(1 - z\overline{\zeta})^{\frac{1}{h}}} \, d\mu_h(\zeta) = (\frac{1}{h} - 1) \int_{\mathbb{D}} f(\zeta) \left(\frac{1 - \zeta\overline{\zeta}}{1 - z\overline{\zeta}} \right)^{\frac{1}{h}} d\mu(\zeta).$$

Given a function $a(z) \in L_\infty(\mathbb{D})$, the Toeplitz operator $T^{(h)}_a$ with symbol a is defined on $\mathcal{A}^2_h(\mathbb{D})$ as follows

$$T^{(h)}_a : \varphi \in \mathcal{A}^2_h(\mathbb{D}) \longmapsto B^{(h)}_{\mathbb{D}}(a\varphi) \in \mathcal{A}^2_h(\mathbb{D}).$$

In what follows we will often consider the C^*-algebra generated by Toeplitz operators $T^{(h)}_a$ whose symbols a belong to a certain subset $\mathcal{A}(\mathbb{D})$ (linear space, subalgebra) of $L_\infty(\mathbb{D})$. Worth mentioning that the composition $T^{(h)}_{a_1} T^{(h)}_{a_2}$ of two such Toeplitz operators is usually not anymore a Toeplitz operator $T^{(h)}_a$ with some $a \in \mathcal{A}(\mathbb{D})$.

We consider the unit disk $\mathbb{D}$ as the hyperbolic plane equipped with the standard hyperbolic metric

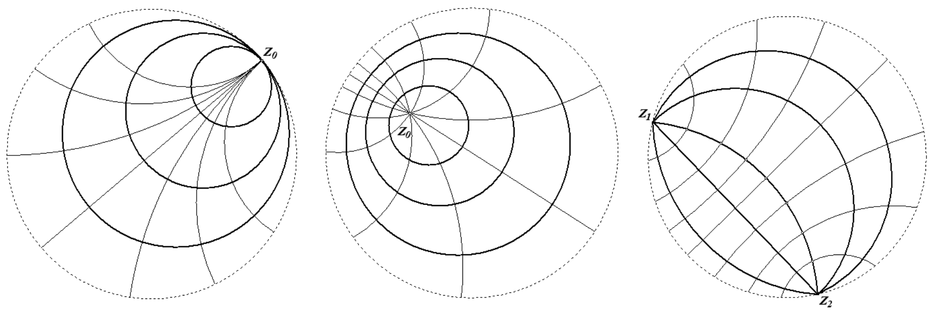
$$ds^2 = \frac{1}{\pi} \frac{dx^2 + dy^2}{(1 - (x^2 + y^2))^2}.$$

Recall that a geodesic, or a hyperbolic straight line, on $\mathbb{D}$ is a part of an Euclidean circle or of a straight line orthogonal to the boundary of $\mathbb{D}$.

Each pair of geodesics, say L_1 and L_2, determines (see, for example, [**1**]) a geometrically defined object, a one-parameter family $\mathcal{P}$ of geodesics, which is called the *pencil* defined by L_1 and L_2. Each pencil has an associated family $\mathcal{C}$ of lines, called *cycles*, which are the orthogonal trajectories to geodesics forming the pencil.

The pencil $\mathcal{P}$ defined by L_1 and L_2 is called

(1) *parabolic* if L_1 and L_2 are parallel (and tend to the same point $z_0 \in \partial\mathbb{D}$), in this case $\mathcal{P}$ is the set of all geodesics parallel to L_1 and L_2, and the cycles are called *horocycles*;
(2) *elliptic* if L_1 and L_2 are intersecting (at a point $z_0 \in \mathbb{D}$), in this case $\mathcal{P}$ is the set of all geodesics passing through the common point of L_1 and L_2;
(3) *hyperbolic* if L_1 and L_2 are disjoint, in this case $\mathcal{P}$ is the set of all geodesics orthogonal to the unique common orthogonal geodesic (with endpoints z_1, $z_2 \in \partial\mathbb{D}$) of L_1 and L_2, and the cycles are called *hypercycles*.



Parabolic, elliptic and hyperbolic pencils.

In the above figure, illustrating possible pencils, the cycles are drawn in bold lines.

The following classification theorem has been proved in [**19, 20**] for the classical (weightless) Bergman space, and in [**7, 8, 9**] for all weighted Bergman spaces.

THEOREM 2.1. *Given a pencil $\mathcal{P}$ of geodesics, consider the set of L_∞-symbols which are constant on corresponding cycles. The C^*-algebra generated by Toeplitz operators with such symbols is commutative on each weighted Bergman space $\mathcal{A}^2_h(\mathbb{D})$.*

3. Sketch of the proof of Theorem 2.1

First of all, each pencil of geodesics of a certain type, parabolic, elliptic, or hyperbolic, can be transformed using an appropriate Möbius transformation to a certain model case.

As the model case for elliptic pencils we consider the pencil where the geodesics intersect at the origin. In this case geodesics are diameters, cycles are concentric Euclidean circles centered at the origin, and symbols, constant on cycles, are just radial functions.

For the model cases for parabolic and hyperbolic pencils we use Möbius transformations from the unit disk onto the upper half-plane Π. The geodesics of the model parabolic pencil on Π are the Euclidean half-line orthogonal to the real axis, horocycles are the Euclidean straight lines parallel to the real axis, and symbols, constant on horocycles, are the functions depending only on $y = \operatorname{Im} z$.

The geodesics of the model hyperbolic pencil on Π are the upper half-circles centered at the origin, hypercycles are the Euclidean half-lines having the common starting point at the origin, and the symbols, constant on hypercycles are the homogeneous functions of the zero order, i.e., the functions depending only on the angle $\theta \in (0, \pi)$ of the polar coordinates in Π.

Both the Bergman space and the Bergman projection are Möbius invariant, thus it is sufficient to prove Theorem 2.1 only for the model cases of pencils. We sketch the proof for the elliptic model case only considering the Bergman spaces parameterized by λ.

The Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$ can be alternatively characterized as the set of all $L_2(\mathbb{D}, d\mu_\lambda)$ functions which satisfy the Cauchy-Riemann equation

$$\frac{\partial}{\partial \overline{z}} f(z) = 0.$$

Passing to the polar coordinates in $\mathbb{D}$ ($z = rt$, $r \in [0,1)$, $t \in S^1$), we have

$$L_2(\mathbb{D}, d\mu_\lambda) = L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr) \otimes L_2(S^1, \frac{dt}{it}),$$

and

$$\frac{\partial}{\partial \overline{z}} = \frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t}\right).$$

Introduce the unitary operator

$$U_1 = I \otimes \mathcal{F} : \; L_2(\mathbb{D}, d\mu_\lambda) \longrightarrow L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr) \otimes l_2,$$

where the discrete Fourier transform $\mathcal{F} : L_2(S^1, \frac{dt}{it}) \to l_2$ is given by

$$\mathcal{F} : \; f \longmapsto c_n = \frac{1}{\sqrt{2\pi}} \int_{S^1} f(t)\, t^{-n} \frac{dt}{it}, \quad n \in \mathbb{Z}.$$

It is easy to check that

$$(I \otimes \mathcal{F})\frac{t}{2}\left(\frac{\partial}{\partial r} - \frac{t}{r}\frac{\partial}{\partial t}\right)(I \otimes \mathcal{F}^{-1})\{c_n(r)\}_{n\in\mathbb{Z}} = \left\{\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{n-1}{r}\right)c_{n-1}(r)\right\}_{n\in\mathbb{Z}}.$$

Thus the image of the Bergman space $\mathcal{A}^2_{1,\lambda} = U_1(\mathcal{A}^2_\lambda(\mathbb{D}))$ can be described as the (closed) subspace of

$$L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr) \otimes l_2 = l_2(L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr)),$$

which consists of all sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ satisfying the equations

$$\frac{1}{2}\left(\frac{\partial}{\partial r} - \frac{n}{r}\right)c_n(r) = 0, \quad n \in \mathbb{Z}.$$

Their general solutions have obviously the form

$$c_n(r) = c'_n\, r^n, \quad n \in \mathbb{Z}.$$

Each function $c_n(r) = c'_n\, r^n$ has to be in $L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr)$, which implies that $c_n(r) \equiv 0$, for all $n < 0$. That is, the space $\mathcal{A}^2_1$ coincides with the space of all two-sided sequences $\{c_n(r)\}_{n\in\mathbb{Z}}$ with

$$c_n(r) = \begin{cases} \sqrt{2}\,\alpha_{n,\lambda} c_n r^n, & \text{if } n \in \mathbb{Z}_+ \\ 0, & \text{if } n \in \mathbb{Z} \setminus \mathbb{Z}_+ \end{cases},$$

where the normalizing constants

$$\alpha_{n,\lambda} = \left(\frac{\lambda+1}{\pi} B(n+1, \lambda+1)\right)^{-\frac{1}{2}} = \left(\frac{\pi\,\Gamma(n+2+\lambda)}{n!\,\Gamma(2+\lambda)}\right)^{\frac{1}{2}}, \qquad n \in \mathbb{Z}_+$$

are selected so that

$$\|\{c_n(r)\}_{n\in\mathbb{Z}}\| = \left(\sum_{n\in\mathbb{Z}_+} |c_n|^2\right)^{1/2} = \|\{c_n\}_{n\in\mathbb{Z}_+}\|_{l_2}.$$

For each $n \in \mathbb{Z}_+$ introduce the unitary operator

$$u_{n,\lambda} : L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr) \longrightarrow L_2([0,1), rdr)$$

by the rule

$$(u_{n,\lambda} f)(r) = \alpha^{-1}_{n,\lambda}\, \omega^{-n}_{n,\lambda}(r)\, f(\omega_{n,\lambda}(r)),$$

where $r = \omega_{n,\lambda}(s)$ is the inverse function to the function

$$\begin{aligned}\sigma_{n,\lambda}(r) &= \alpha_{n,\lambda}\left(\frac{\lambda+1}{\pi}\int_0^r u^{2n+1}(1-u^2)^\lambda du\right)^{\frac{1}{2}}\\ &= \alpha_{n,\lambda}\left(\frac{\lambda+1}{\pi}B_{r^2}(n+1,\lambda+1)\right)^{\frac{1}{2}}\\ &= I_{r^2}(n+1,\lambda+1)^{\frac{1}{2}},\end{aligned}$$

where the incomplete B-function B_x and the function I_x are given by formulas 8.391 and 8.392 in [**6**].

Finally, define the unitary operator

$$U_{2,\lambda} : l_2(L_2([0,1), \frac{\lambda+1}{\pi}(1-r^2)^\lambda\, rdr)) \longrightarrow l_2(L_2([0,1), rdr)) = L_2([0,1), rdr) \otimes l_2$$

as follows

$$U_{2,\lambda} : \ \{c_n(r)\}_{n\in\mathbb{Z}} \ \longmapsto \ \{(u_{|n|,\lambda}c_n)(r)\}_{n\in\mathbb{Z}}.$$

Let $\ell_0(r) = \sqrt{2}$, and let L_0 be the one-dimensional subspace of $L_2([0,1), rdr)$ generated by $\ell_0(r)$. The one-dimensional projection P_0 of $L_2([0,1), rdr)$ onto L_0 has the form

$$(P_0 f)(r) = \langle f, \ell_0\rangle \cdot \ell_0 = \sqrt{2}\int_0^1 f(\rho)\,\sqrt{2}\rho\, d\rho.$$

Denote by l_2^+ the subspace of (two-sided) l_2, consisting of all sequences $\{c_n\}_{n\in\mathbb{Z}}$, such that $c_n = 0$ for all $n \in \mathbb{Z}\setminus\mathbb{Z}_+$, and let p^+ be the orthogonal projection of l_2 onto l_2^+.

Theorem 3.1. *The unitary operator $U_\lambda = U_{2,\lambda}U_1$ gives an isometric isomorphism of the space $L_2(\mathbb{D}, d\mu_\lambda)$ onto $L_2([0,1), rdr) \otimes l_2$ under which*

(1) *the wighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{D})$ is mapped onto $L_0 \otimes l_2^+$*

$$U_\lambda : \mathcal{A}^2_\lambda(\mathbb{D}) \longrightarrow L_0 \otimes l_2^+,$$

where L_0 is the one-dimensional subspace of $L_2([0,1), rdr)$, generated by $\ell_0(r) = \sqrt{2}$,

(2) *the weighted Bergman projection $B^{(\lambda)}_{\mathbb{D}}$ is unitary equivalent to the following one*

$$U_\lambda\, B^{(\lambda)}_{\mathbb{D}}\, U_\lambda^{-1} = P_0 \otimes p^+,$$

where P_0 is the one-dimensional projection of $L_2([0,1), rdr)$ onto L_0.

Introduce now the isometric imbedding

$$R_0 : l_2^+ \longrightarrow L_2([0,1), rdr) \otimes l_2$$

given by

$$R_0 : \ \{c_n\}_{n\in\mathbb{Z}_+} \ \longmapsto \ \ell_0(r)\{\chi_+(n)c_n\}_{n\in\mathbb{Z}},$$

where we extend the sequence $\{c_n\}_{n\in\mathbb{Z}_+}$ to an element of l_2 setting $c_n = 0$ for negative indices $n < 0$.

The operator $R_\lambda = R_0^* U_\lambda$ maps the space $L_2(\mathbb{D}, d\mu_\lambda)$ onto l_2^+, and the restriction

$$R_\lambda|_{\mathcal{A}^2_\lambda(\mathbb{D})} : \mathcal{A}^2_\lambda(\mathbb{D}) \longrightarrow l_2^+$$

is an isometric isomorphism. The adjoint operator

$$R_\lambda^* = U_\lambda^* R_0 : l_2^+ \longrightarrow \mathcal{A}^2_\lambda(\mathbb{D}) \subset L_2(\mathbb{D}, d\mu_\lambda)$$

is an isometric isomorphism of l_2^+ onto the subspace $\mathcal{A}^2_\lambda(\mathbb{D})$ of the space $L_2(\mathbb{D}, d\mu_\lambda)$.

Moreover we have

$$\begin{aligned} R_\lambda\, R_\lambda^* = I \quad &: \quad l_2^+ \longrightarrow l_2^+, \\ R_\lambda^* R_\lambda = B_{\mathbb{D}}^{(\lambda)} \quad &: \quad L_2(\mathbb{D}, d\mu_\lambda) \longrightarrow \mathcal{A}^2_\lambda(\mathbb{D}). \end{aligned}$$

Given a radial function $a = a(r) \in L_\infty(0,1)$, consider the Toeplitz operator

$$T_a : \varphi \in \mathcal{A}^2_\lambda(\mathbb{D}) \longmapsto B_{\mathbb{D}}^{(\lambda)} a\varphi \in \mathcal{A}^2_\lambda(\mathbb{D}).$$

Theorem 3.2. *For any $a = a(r) \in L_\infty[0,1)$, the Toeplitz operator T_a acting on $\mathcal{A}^2_\lambda(\mathbb{D})$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I = R_\lambda T_a R_\lambda^*$, acting on l_2^+. The sequence $\gamma_{a,\lambda} = \{\gamma_{a,\lambda}(n)\}_{n\in\mathbb{Z}_+}$ is as follows*

$$\gamma_{a,\lambda}(n) = \frac{1}{B(n+1,\lambda+1)} \int_0^1 a(\sqrt{r}) r^n (1-r)^\lambda dr, \qquad n \in \mathbb{Z}_+. \tag{3.1}$$

Corollary 3.3. *The C^*-algebra $\mathcal{T}_\lambda$ generated by all Toeplitz operators T_a with symbols $a = a(r) \in L_\infty[0,1)$ is commutative and is isometrically imbedded to l_∞. The isometric imbedding τ_λ is generated by the mapping*

$$\tau_\lambda : \ T_a \longmapsto \gamma_{a,\lambda}.$$

The proofs for the other two cases, parabolic and hyperbolic, are rather similar. But, instead of the discrete Fourier transform used in the proof for elliptic case, the Fourier transform and the Mellin transform are used for parabolic and hyperbolic cases, respectively. For these remaining cases we have

Theorem 3.4. *Given either parabolic or hyperbolic model pencil and a symbol $a \in L_\infty(D)$, constant on corresponding cycles, the Toeplitz operator $T_a^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda} I$, where*
in the parabolic case: $a = a(y)$, $y \in \mathbb{R}_+$, $\quad \gamma_{a,\lambda} I : L_2(\mathbb{R}_+) \to L_2(\mathbb{R}_+)$, and

$$\gamma_{a,\lambda}(x) = \frac{x^{\lambda+1}}{\Gamma(\lambda+1)} \int_0^\infty a(y/2) y^\lambda e^{-xy} dy, \qquad x \in \mathbb{R}_+,$$

in the hyperbolic case: $a = a(\theta)$, $\theta \in (0,\pi)$, $\quad \gamma_{a,\lambda} I : L_2(\mathbb{R}) \to L_2(\mathbb{R})$, and

$$\gamma_{a,\lambda}(\xi) = 2^\lambda \ (\lambda+1) \frac{|\Gamma(\frac{\lambda+2}{2} + i\xi)|^2}{\pi\, \Gamma(\lambda+2)\, e^{\pi\xi}} \int_0^\pi a(\theta)\, e^{-2\xi\theta} \sin^\lambda \theta \ d\theta, \qquad \xi \in \mathbb{R}.$$

4. Three-term asymptotic expansion formula

To get an inverse statement to Theorem 2.1 we will use the familiar Berezin quantization procedure on the unit disk (see, for example, [**3, 4**]).

For each function $a = a(z) \in C^\infty(\mathbb{D})$ consider the family of Toeplitz operators $T_a^{(h)}$ with (anti-Wick) symbol a acting on $\mathcal{A}^2_h(\mathbb{D})$, for $h \in (0,1)$. The Wick symbols of the Toeplitz operator $T_a^{(h)}$ has the form

$$\widetilde{a}_h(z,\overline{z}) = (\frac{1}{h} - 1) \int_{\mathbb{D}} a(\zeta) \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})} \right)^{\frac{1}{h}} d\mu(\zeta),$$

and the star product of Wick symbols is defined as follows

$$(\widetilde{a}_h \star \widetilde{b}_h)(z,\overline{z}) = (\frac{1}{h} - 1) \int_{\mathbb{D}} \widetilde{a}_h(z,\overline{\zeta})\, \widetilde{b}_h(\zeta,\overline{z}) \left(\frac{(1-|z|^2)(1-|\zeta|^2)}{(1-z\overline{\zeta})(1-\zeta\overline{z})} \right)^{\frac{1}{h}} d\mu(\zeta).$$

The correspondence principle is given by

$$\begin{aligned}
\widetilde{a}_h(z,\overline{z}) &= a(z,\overline{z}) + O(\hbar), \\
(\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h)(z,\overline{z}) &= i\hbar\,\{a,b\} + O(\hbar^2).
\end{aligned}$$

The information which follows from the last formula is insufficient for our purposes. To get a desired result we need the three-term asymptotic expansion formula of the commutator of two Wick symbols, which is given by the next theorem.

Theorem 4.1. *For any pair $a = a(z,\overline{z})$ and $b = b(z,\overline{z})$ of six times continuously differentiable functions the following three-term asymptotic expansion formula holds*

$$\begin{aligned}
\widetilde{a}_h \star \widetilde{b}_h - \widetilde{b}_h \star \widetilde{a}_h &= i\hbar\,\{a,b\} + i\frac{\hbar^2}{4}\left(\Delta\{a,b\} + \{a,\Delta b\} + \{\Delta a, b\} + 8\pi\{a,b\}\right) \\
&\quad + i\frac{\hbar^3}{24}\big[\{\Delta a,\Delta b\} + \{a,\Delta^2 b\} + \{\Delta^2 a, b\} + \Delta^2\{a,b\} \\
&\quad + \Delta\{a,\Delta b\} + \Delta\{\Delta a, b\} + 28\pi\left(\Delta\{a,b\} + \{a,\Delta b\} + \{\Delta a, b\}\right) \\
&\quad + 96\pi^2\{a,b\}\big] + o(\hbar^3),
\end{aligned}$$

where $\hbar = \frac{h}{2\pi}$, and the Poisson bracket and the Laplace-Beltrami operator are given by

$$\begin{aligned}
\{a,b\} &= 2\pi i(1 - z\overline{z})^2\left(\frac{\partial a}{\partial z}\frac{\partial b}{\partial \overline{z}} - \frac{\partial a}{\partial \overline{z}}\frac{\partial b}{\partial z}\right), \\
\Delta &= 4\pi(1 - z\overline{z})^2\frac{\partial^2}{\partial z \partial \overline{z}}.
\end{aligned}$$

Corollary 4.2. *Let $\mathcal{A}(\mathbb{D})$ be a subalgebra of $C^\infty(\mathbb{D})$ such that for each $h \in (0,1)$ the Toeplitz operator algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ is commutative. Then for all a, $b \in \mathcal{A}(\mathbb{D})$ we have*

$$\{a,b\} = 0, \tag{4.1}$$

$$\{a,\Delta b\} + \{\Delta a, b\} = 0, \tag{4.2}$$

$$\{\Delta a,\Delta b\} + \{a,\Delta^2 b\} + \{\Delta^2 a, b\} = 0. \tag{4.3}$$

5. Consequences of (4.1), (4.2), and (4.3)

As we consider the C^*-algebra generated by Toeplitz operators, we can always assume, without loss of generality, that our set of symbols is a linear space closed under complex conjugation and containing the function $e(z) \equiv 1$.

Note that there is a trivial case of such a sitiation having in fact no connection with specific properties of Toeplitz operators. Each C^*-algebra with identity (Toeplitz operators with the symbol $e(z)$) generated by a self-adjoint element (Toeplitz operator with a real valued symbol $a = a(z)$) is obviously commutative. The set of generating symbols here is quite restricted, and coincides with the two dimensional linear space generated by $e(z)$ and $a(z)$. In what follows we exclude this obvious case from our consideration.

To underline the geometric nature of symbol classes which generate the commutative C^*-algebras of Toeplitz operators we have considered bounded measurable symbols in Theorem 2.1. This also agrees with the desire for such (commutative) algebras to be, in a sense, maximal. Note that the arguments used in the proof do not require any assumption on smoothness properties of symbols. The same result (commutativity of Toeplitz operator C^*-algebra) remains valid for any linear subspace of L_∞-symbols (constant on cycles). Moreover, we can start with a much more restricted set of symbols (say, smooth symbols only) and extend them furthermore to all L_∞-symbols by means of uniform and strong operator limits of sequences of Toeplitz operators.

Let $\mathcal{A}(\mathbb{D})$ be a linear space of (smooth) functions. Denote by $\mathcal{T}(\mathcal{A}(\mathbb{D})) = \{\mathcal{T}_h(\mathcal{A}(\mathbb{D}))\}_h$ the family of C^*-algebras $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ generated by Toeplitz operators with symbols from $\mathcal{A}(\mathbb{D})$ and acting on the weighted Bergman spaces $\mathcal{A}_h^2(\mathbb{D})$.

To introduce our symbol classes we need the notion of the jet of a function (see, for example, [**15, 17**]). Given two complex valued smooth functions f and g defined in a neighborhood of a point $z \in \mathbb{D}$, we say that they have the same jet of order k at z if their real partial derivatives at z up to order k are equal. It is easy to see that such relation does not depend on the coordinate system and that it defines an equivalence relation. The corresponding equivalence class of a function f at z is denoted by $j_z^k(f)$ and is called the k-th order jet of f at z. Furthermore, given a complex vector space $\mathcal{A}(\mathbb{D})$ of smooth functions, we denote with $J_z^k(\mathcal{A}(\mathbb{D}))$ the space of k-jets at z of the elements in $\mathcal{A}(\mathbb{D})$. We observe that $J_z^k(\mathcal{A}(\mathbb{D}))$ is a finite dimensional complex vector space.

In what follows, for a differentiable function $f : \mathbb{D} \to \mathbb{C}$ we will say that $z \in \mathbb{D}$ is a nonsingular point of f if $df_z \neq 0$.

The symbol classes that we are considering are given in the next definition.

Definition 5.1. Let $\mathcal{A}(\mathbb{D})$ be a complex vector space of smooth functions. We will say that $\mathcal{A}(\mathbb{D})$ is k-rich if it is closed under complex conjugation and the following conditions are satisfied:

(i) there is a finite set S such that for every $z \in \mathbb{D} \setminus S$ at least one element of $\mathcal{A}(\mathbb{D})$ is nonsingular at z,

(ii) for every point $z \in \mathbb{D} \setminus S$ and $l = 0, \dots, k$, the space of jets $J_z^l(\mathcal{A}(\mathbb{D}))$ has complex dimension at least $l + 1$.

Observe that k-richness implies l-richness for $l \leq k$. The following result ensures that k-richness, for each $k \geq 2$, excludes from consideration commutative Toeplitz C^*-algebras with identity generated by a single self-adjoint Toeplitz operator.

Lemma 5.2. *Let $\mathcal{A}(\mathbb{D})$ be a 2-rich space of smooth functions. Then, there is no open set V in $\mathbb{D}$ such that the restriction $\mathcal{A}(\mathbb{D})|_V$ is generated by a single real valued function $a \in \mathcal{A}(\mathbb{D})$ and $e(z) \equiv 1$.*

As the set $\mathcal{A}(\mathbb{D})$ is closed under the complex conjugation, it is sufficient to consider the conditions (4.1), (4.2), and (4.3) for real valued functions only. Recall that each real valued function $a \in \mathcal{A}(\mathbb{D})$, nonsingular in some open set, has in this set two systems of mutually orthogonal smooth lines, the system of level lines and the system of gradient lines.

Given any such a pair, a function a and an open set U, it is easy to see that the two above systems of lines can be parameterized to be a new orthogonal coordinate system (u, v) in U. The level lines and the gradient lines of the function a in the coordinates (u, v) are given respectively as

$$u = u_0 = \text{const} \qquad \text{and} \qquad v = v_0 = \text{const}\,.$$

Thus, in particular, we have $a = a(u) = a(u(x, y))$.

The coordinate systems (u, v) and (x, y) are connected by

$$u = u(x, y), \quad v = v(x, y), \qquad \text{or} \qquad x = x(u, v), \quad y = y(u, v),$$

with

$$D = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u} \neq 0,$$

and the orthogonality of the coordinate system (u, v) is equivalent to

$$\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} \equiv 0.$$

In the coordinates (u, v) the metric, the symplectic form, and the Poisson brackets have respectively the following form

$$ds^2 = \widetilde{g}_{11}(u, v)du^2 + \widetilde{g}_{22}(u, v)dv^2,$$

where

$$\widetilde{g}_{11} = g(x, y)\left[\left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2\right], \qquad \widetilde{g}_{22} = g(x, y)\left[\left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2\right],$$

with $g = g(x, y) = \pi^{-1}(1 - (x^2 + y^2))^{-2}$, and

$$\begin{aligned} \omega &= g(x, y)\, D\, du \wedge dv, \\ \{f_1, f_2\} &= g^{-1}(x, y)\, D \left(\frac{\partial f_1}{\partial v}\frac{\partial f_2}{\partial u} - \frac{\partial f_1}{\partial u}\frac{\partial f_2}{\partial v}\right). \end{aligned} \tag{5.1}$$

The geometric information contained in the first term of asymptotic expansion of a commutator, or equivalently in the condition (4.1), is given by the next lemma.

LEMMA 5.3. *Let $\mathcal{A}(\mathbb{D})$ be a 2-rich space of smooth functions which generates for each $h \in (0, 1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators. Then all real valued functions in $\mathcal{A}(\mathbb{D})$ have (globally) the same set of level lines and the same set of gradient lines.*

PROOF. Fix a real valued function $a \in \mathcal{A}(\mathbb{D})$ and the local orthogonal coordinate system as above. For any other (real valued) function $b \in \mathcal{A}(\mathbb{D})$ condition (4.1) implies

$$\{a, b\} = -g^{-1} D\, a'(u) \frac{\partial b}{\partial v} \equiv 0,$$

or

$$\frac{\partial b}{\partial v} \equiv 0,$$

that is the function b has (in U_0) the *same* level lines, and thus has the *same* gradient lines as the function a. □

The Laplace-Beltrami operator in the coordinates (u, v) has the form (see, for example, [**16**], p. 87)

$$\Delta = \widetilde{g}^{11}\left(\frac{\partial^2}{\partial u^2} - \widetilde{\Gamma}^1_{11}\frac{\partial}{\partial u} - \widetilde{\Gamma}^2_{11}\frac{\partial}{\partial v}\right) + \widetilde{g}^{22}\left(\frac{\partial^2}{\partial v^2} - \widetilde{\Gamma}^1_{22}\frac{\partial}{\partial u} - \widetilde{\Gamma}^2_{22}\frac{\partial}{\partial v}\right),$$

where the matrix $(\widetilde{g}^{ij})$ is inverse to $(\widetilde{g}_{ij})$ and $\widetilde{\Gamma}^k_{ij}$ are the Schwarz-Christoffel symbols on (u, v).

For any function $c = c(u) \in \mathcal{A}(\mathbb{D})$ we have

$$\Delta c = c''\widetilde{g}^{11} - c'\left(\widetilde{g}^{11}\widetilde{\Gamma}^1_{11} + \widetilde{g}^{22}\widetilde{\Gamma}^1_{22}\right). \tag{5.2}$$

Vanishing of the second term of asymptotic in a commutator, or equivalently the condition (4.2), leads to the following theorem.

THEOREM 5.4. *Let $\mathcal{A}(\mathbb{D})$ be a 2-rich space of smooth functions which generates for each $h \in (0,1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators. Then the common gradient lines of all real valued functions in $\mathcal{A}(\mathbb{D})$ are geodesics in the hyperbolic geometry of the unit disk $\mathbb{D}$.*

PROOF. Given two real valued functions $a = a(u)$, $b = b(u) \in \mathcal{A}(\mathbb{D})$. the condition (4.2) is equivalent to

$$\begin{aligned} 0 &\equiv a' \cdot \frac{\partial \Delta b}{\partial v} - b' \cdot \frac{\partial \Delta a}{\partial v} \\ &= (a'b'' - b'a'')\,\frac{\partial \widetilde{g}^{11}}{\partial v} - (a'b' - b'a')\,\frac{\partial}{\partial v}\left(\widetilde{g}^{11}\widetilde{\Gamma}^1_{11} + \widetilde{g}^{22}\widetilde{\Gamma}^1_{22}\right) \\ &= (a'b'' - b'a'')\,\frac{\partial \widetilde{g}^{11}}{\partial v}. \end{aligned}$$

Note, that vanishing of $a'b'' - b'a''$ in an open subset of U is equivalent to the property that in this subset one of the functions, a or b, is a linear combination of the other and $e(z) \equiv 1$, which is impossible by Lemma 5.2. By Lemma 5.2 we can change, if necessary, in different parts of U the functions a and b from $\mathcal{A}(\mathbb{D})$ in order to have $a'b'' - b'a'' \neq 0$; then

$$\frac{\partial \widetilde{g}^{11}}{\partial v} = -\widetilde{g}^2_{11}\,\frac{\partial \widetilde{g}_{11}}{\partial v} \equiv 0, \tag{5.3}$$

or

$$\frac{\partial}{\partial v}\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle \equiv 0, \tag{5.4}$$

where $\langle X_1, X_2 \rangle = ds^2(X_1, X_2)$ is the inner product of the vector fields X_1 and X_2.

Let γ be a gradient line. Its Frenet frame (e_1, e_2) is given by

$$e_1 = \left\| \frac{\partial}{\partial u} \right\|^{-1} \frac{\partial}{\partial u}, \qquad e_2 = \left\| \frac{\partial}{\partial v} \right\|^{-1} \frac{\partial}{\partial v}.$$

By [**14**], page 78, the geodesic curvature $\kappa_\gamma(u)$ of γ is calculated as follows

$$\kappa_\gamma(u) = \left\| \frac{\partial}{\partial u} \right\|^{-1} \left\langle e_2, \nabla_{\frac{\partial}{\partial u}} e_1 \right\rangle.$$

Using standard properties of the connection ∇, we have

$$\nabla_{\frac{\partial}{\partial u}} e_1 = \left(\frac{\partial}{\partial u} \left\| \frac{\partial}{\partial u} \right\|^{-1} \right) \cdot \frac{\partial}{\partial u} + \left\| \frac{\partial}{\partial u} \right\|^{-1} \cdot \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}.$$

Thus

$$\kappa_\gamma(u) = \left\| \frac{\partial}{\partial u} \right\|^{-1} \left\langle e_2, \nabla_{\frac{\partial}{\partial u}} e_1 \right\rangle = \left\| \frac{\partial}{\partial u} \right\|^{-2} \left\| \frac{\partial}{\partial v} \right\|^{-1} \left\langle \frac{\partial}{\partial v}, \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u} \right\rangle. \tag{5.5}$$

By the Koszul formula (see, for example, [**16**], page 61) we have

$$\begin{aligned} 2\left\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle &= \frac{\partial}{\partial u}\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle + \frac{\partial}{\partial u}\left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right\rangle - \frac{\partial}{\partial v}\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle \\ &- \left\langle \frac{\partial}{\partial u}, \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] \right\rangle + \left\langle \frac{\partial}{\partial u}, \left[\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right] \right\rangle + \left\langle \frac{\partial}{\partial v}, \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right] \right\rangle, \end{aligned}$$

where $[X_1, X_2]$ is the commutator of the vector fields X_1 and X_2.

Taking into account that

$$\left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 0, \qquad \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right] = - \left[\frac{\partial}{\partial v}, \frac{\partial}{\partial u} \right] = 0, \qquad \left[\frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right] = 0,$$

we have

$$\left\langle \nabla_{\frac{\partial}{\partial u}} \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = -\frac{1}{2} \frac{\partial}{\partial v} \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle . \tag{5.6}$$

Finally, from (4.1), (5.4), (5.5), and (5.6) it follows that $\kappa_\gamma \equiv 0$, and thus (see, for example, [**14**], Proposition 4.3.2) the system of gradient lines consists of geodesics. □

Vanishing of the third term of asymptotic in a commutator, or equivalently the condition (4.3), implies the following theorem.

Theorem 5.5. *Let $\mathcal{A}(\mathbb{D})$ be a 3-rich vector space of smooth functions $\mathcal{A}(\mathbb{D})$ which generates for each $h \in (0,1)$ the commutative C^*-algebra $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ of Toeplitz operators. Then the common level lines of all real valued functions in $\mathcal{A}(\mathbb{D})$ are cycles.*

The next theorem provides a geometric characterization of the real valued functions on $\mathbb{D}$ whose gradient lines define a pencil of geodesics.

Theorem 5.6. *A nonconstant C^3 real valued function a in $\mathbb{D}$ defines a pencil if and only if the following two conditions are satisfied:*

(i) *The gradient lines of a are geodesics.*
(ii) *Each level line of a is a cycle.*

Lemma 5.3, Theorem, 5.4, Corollary 5.5, and Theorem 5.6 lead directly to the following result.

Corollary 5.7. *Let $\mathcal{A}(\mathbb{D})$ be a 3-rich vector space of smooth functions such that $\mathcal{T}_h(\mathcal{A}(\mathbb{D}))$ is commutative for each $h \in (0,1)$. Then there exists a pencil $\mathcal{P}$ of geodesics in $\mathbb{D}$ such that all functions in $\mathcal{A}(\mathbb{D})$ are constant on the cycles of $\mathcal{P}$.*

Now the main result reads as follows.

Theorem 5.8. *Let $\mathcal{A}(\mathbb{D})$ be a 3-rich vector space of smooth functions. Then the following statements are equivalent:*

(i) *there is a pencil $\mathcal{P}$ of geodesics in $\mathbb{D}$ such that all functions in $\mathcal{A}(\mathbb{D})$ are constant on the cycles of $\mathcal{P}$;*
(ii) *the C^*-algebra generated by Toeplitz operators with $\mathcal{A}(\mathbb{D})$-symbols is commutative on each weighted Bergman space $\mathcal{A}^2_h(\mathbb{D})$, $h \in (0,1)$.*

6. Spectra of Toeplitz operators, continuous symbols

Let D be either the unit disk $\mathbb{D}$, or the upper half-plane Π in $\mathbb{C}$. We consider the weighted Bergman space $\mathcal{A}^2_\lambda(D)$, where the measure $d\mu_{D,\lambda}$ is given correspondingly by

$$d\mu_{\mathbb{D},\lambda}(z) = \frac{\lambda+1}{\pi}\,(1-|z|^2)^\lambda\,dxdy, \qquad \text{or} \qquad d\mu_{\Pi,\lambda}(z) = \frac{\lambda+1}{\pi}\,(2\mathrm{Im}\,z)^\lambda\,dxdy.$$

Recall that the weighted Bergman projection $B_D^{(\lambda)}$ of $L_2(D, d\mu_{D,\lambda})$ has in either case the form

$$(B_{\mathbb{D}}^{(\lambda)}\varphi)(z) = \int_{\mathbb{D}} \frac{\varphi(\zeta)}{(1 - z\overline{\zeta})^{\lambda+2}}\, d\mu_{\mathbb{D},\lambda}(\zeta),$$

or

$$(B_{\Pi}^{(\lambda)}\varphi)(z) = i^{\lambda+2} \int_{\Pi} \frac{\varphi(\zeta)}{(z - \overline{\zeta})^{\lambda+2}}\, d\mu_{\Pi,\lambda}(\zeta).$$

The *a priori* spectral information for general L_∞-symbols (see, for example, [**2**], [**3**]) says that for each $a \in L_\infty(D)$ and each $\lambda \geq 0$

$$\operatorname{sp} T_a^{(\lambda)} \subset \operatorname{conv}(\operatorname{ess} - \operatorname{Range} a). \tag{6.1}$$

At the same time, given any model pencil (parabolic, elliptic, or hyperbolic) and a symbol $a \in L_\infty(D)$, constant on corresponding cycles, by Theorems 3.2 and 3.4, the Toeplitz operator $T_a^{(\lambda)}$ is unitary equivalent to the multiplication operator $\gamma_{a,\lambda}I$. Thus, for each λ, we have obviously

$$\operatorname{sp} T_a^{(\lambda)} = \overline{M_\lambda(a)}, \qquad \text{where} \quad M_\lambda(a) = \operatorname{Range} \gamma_{a,\lambda}.$$

We will use the following notion of the limit set for a family of subsets in $\mathbb{C}$.

Let E be a subset of $\mathbb{R}$ having $+\infty$ as a limit point, and let for each $\lambda \in E$ there is a set $M_\lambda \subset \mathbb{C}$. Define the set M_∞ as the set of all $z \in \mathbb{C}$ for which there exists a sequence of complex numbers $\{z_n\}_{n\in\mathbb{N}}$ such that

(i) for each $n \in \mathbb{N}$ there exists $\lambda_n \in E$ such that $z_n \in M_{\lambda_n}$,
(ii) $\lim_{n\to\infty} \lambda_n = +\infty$,
(iii) $z = \lim_{n\to\infty} z_n$.

We will write

$$M_\infty = \lim_{\lambda\to+\infty} M_\lambda,$$

and call M_∞ the (partial) limit set of a family $\{M_\lambda\}_{\lambda\in E}$ when $\lambda \to +\infty$.

For the case when E is a discrete set with a unique limit point at infinity, the above notion coincides with the partial limiting set introduced in [**11**], Section 3.1.1. Following the arguments of Proposition 3.5 in [**11**] one can show that

$$M_\infty = \bigcap_{\lambda} \operatorname{clos}\left(\bigcup_{\mu\geq\lambda} M_\mu\right).$$

Note that obviously

$$\lim_{\lambda\to+\infty} M_\lambda = \lim_{\lambda\to+\infty} \overline{M}_\lambda = M_\infty.$$

Theorem 6.1. *Let a be a continuous symbol constant on cycles. Then*

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) = \operatorname{Range} a. \tag{6.2}$$

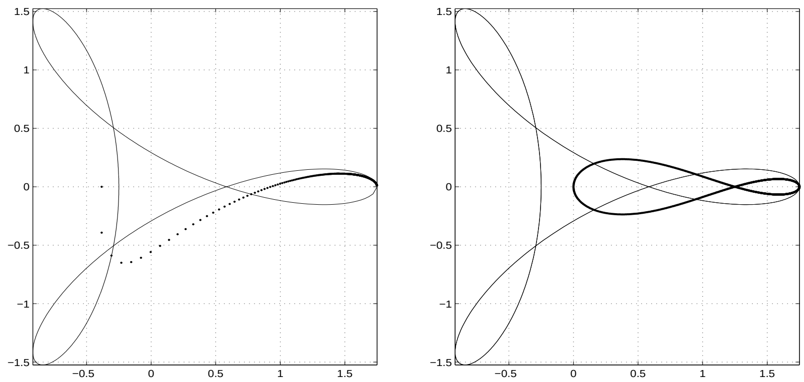
Note, that $\operatorname{Range} a$ coincides with the spectrum $\operatorname{sp} aI$ of the operator of multiplication by $a = a(y)$, thus the another form of (6.2) is

$$\lim_{\lambda\to+\infty} \operatorname{sp} T_a^{(\lambda)} = \operatorname{sp} aI.$$

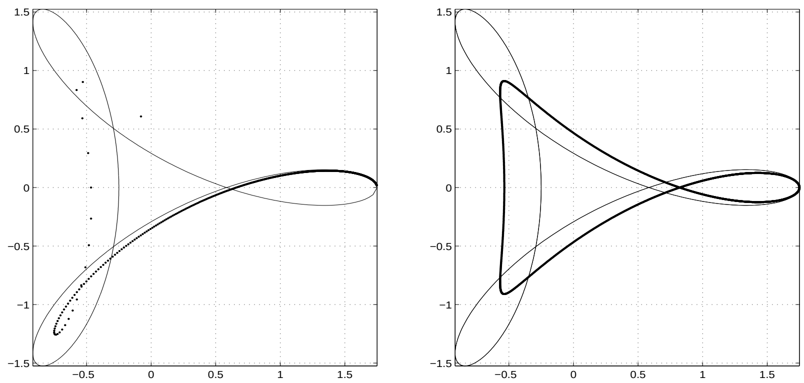
We illustrate the theorem on two continuous symbol (both are hypocycloids)

$$a_1(r) = \frac{3}{4}(r + i\sqrt{1-r^2})^8 + (r - i\sqrt{1-r^2})^4 \qquad \text{and} \qquad a_2(\theta) = \frac{3}{4}e^{4i\theta} + e^{-2i\theta},$$

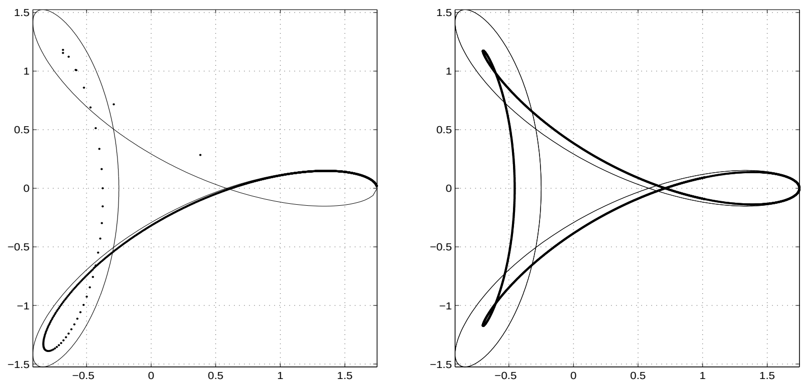
presenting the images of the sequence $\gamma_{a_1,\lambda}(n)$ and the function $\gamma_{a_2,\lambda}(\xi)$ for the following values of λ: 0, 5, 12, and 200.



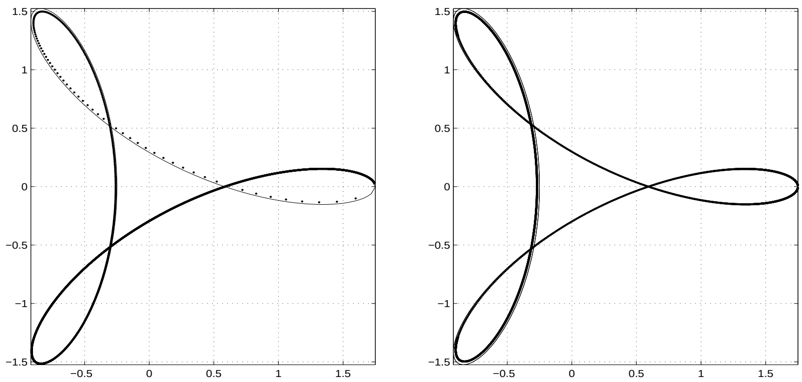
The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 0$.



The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 5$.



The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 12$.



The images of $\gamma_{a_1,\lambda}$ and $\gamma_{a_2,\lambda}$ for $\lambda = 200$.

7. Spectra of Toeplitz operators, piecewise continuous symbols

Let a be a piecewise continuous symbol constant on cycles and having a finite number m of jump points. Denote by $\bigcup_{j=1}^{m} I_j(a)$ the union of the straight line segments connecting the one-sided limit values of a at the jump points. Introduce

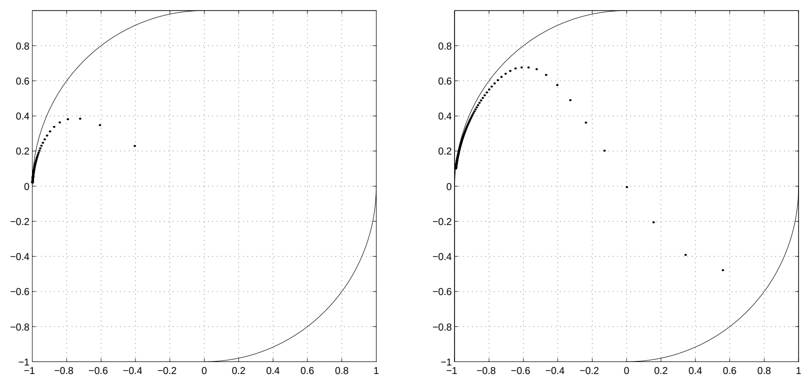
$$\widetilde{R}(a) = \mathrm{Range}\, a \cup \left(\bigcup_{j=1}^{m} I_j(a) \right).$$

THEOREM 7.1. *Let a be a piecewise continuous symbol constant on cycles. Then*

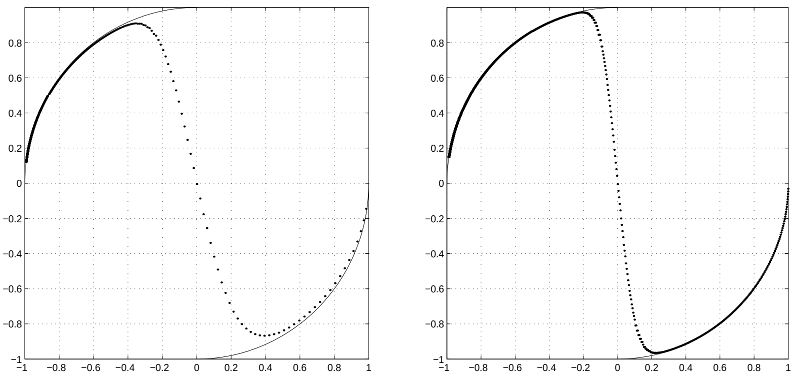
$$\lim_{\lambda \to \infty} \mathrm{sp}\, T_a^{(\lambda)} = M_\infty(a) = \widetilde{R}(a).$$

We illustrate the theorem on the following radial piecewise continuous symbol

$$a(r) = \begin{cases} e^{-i\pi r^2}, & r \in [0, 1/\sqrt{2}], \\ e^{i\pi r^2}, & r \in (1/\sqrt{2}, 1]. \end{cases}$$



The sequence $\gamma_{a,\lambda}$ for $\lambda = 0$ and $\lambda = 4$.



The sequence $\gamma_{a,\lambda}$ for $\lambda = 40$ and $\lambda = 200$.

We note that the appearance of straight line segments which connect the one-sided limit values at the points of discontinuity of symbols, is quite typical in the theory of Toeplitz operators with piecewise continuous symbols acting either on the Hardy, or on the Bergman space (see, for example, [**5, 18**]). We stress the principal difference between our case and the cases just mentioned. In the mentioned case of Toeplitz operators with piecewise continuous symbols the straight line segments appear in the *essential spectrum* of the Toeplitz operator. For the radial symbol, continuous at 1, any Toeplitz operator is a compact perturbation of a multiple of the identity, i.e., $T_{a(r)} = a(1)I + K$, and its essential spectrum consists of a single point $a(1)$ for all λ. For each fixed λ the spectrum of a Toeplitz operator is the union of the *discrete set* (the sequence $\gamma_{a,\lambda}(n)$) with its limit point $a(1)$. The tendency of a straight line segment forming starts appearing for large values of λ, and the straight line segments themselves appear only in the *limit set of spectra*.

For general L_∞-symbols apart from the *a priori* information (6.1) we have obviously

$$\lim_{\lambda\to\infty} \operatorname{sp} T_a^{(\lambda)} = M_\infty(a) \subset \operatorname{conv}(\operatorname{ess}-\operatorname{Range} a).$$

At the same time the collocation of $M_\infty(a)$ inside $\operatorname{conv}(\operatorname{ess}-\operatorname{Range} a)$ may essentially vary. We give a number of examples illustrating possible interrelations between these sets. The examples are given for the parabolic case, but the results are the same for the either case, parabolic, elliptic, or hyperbolic.

Example 7.2. Let $a(y) \in C[0,+\infty]$. Then according to Theorem 6.1,

$$M_\infty(a) = \operatorname{Range} a \ (= \operatorname{ess}-\operatorname{Range} a).$$

Example 7.3. Let

$$a(y/2) = \begin{cases} \alpha_1, & t \in (0,1), \\ \alpha_2, & t \in [1,\infty]. \end{cases}$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$ and $\alpha_1 \neq \alpha_2$. Then according to Theorem 7.1 $M_\infty(a)$ coincides with the straight line segment $[\alpha_1, \alpha_2]$ joining the points α_1 and α_2, whence

$$M_\infty(a) = \operatorname{conv}(\operatorname{ess}-\operatorname{Range} a) \quad (= \operatorname{conv}(\operatorname{Range} a)).$$

Example 7.4. Let

$$a(y/2) = \begin{cases} \alpha_1, & t \in [0,1), \\ \alpha_2, & t \in [1,2), \\ \alpha_3, & t \in [2,\infty], \end{cases}$$

where $\alpha_1, \alpha_2, \alpha_3$ are different points from $\mathbb{C}$. Then by Theorem 7.1 we have

$$M_\infty(a) = [\alpha_1, \alpha_2] \cup [\alpha_2, \alpha_3] \subset \partial\,\mathrm{conv}(\mathrm{Range}\, a).$$

Example 7.5. Let $\alpha_1, \alpha_2, \alpha_3$ be as above, and

$$a(y/2) = \begin{cases} \alpha_1, & t \in [0,1), \\ \alpha_2, & t \in [1,2), \\ \alpha_3, & t \in [2,3), \\ \alpha_1, & t \in [3,\infty]. \end{cases}$$

By Theorem 7.1 the set $M_\infty(a)$ coincides with the triangle having vertices $\alpha_1, \alpha_2, \alpha_3$

$$M_\infty(a) = [\alpha_1, \alpha_2] \cup [\alpha_2, \alpha_3] \cup [\alpha_3, \alpha_4] = \partial\,\mathrm{conv}(\mathrm{Range}\, a).$$

8. Spectra of Toeplitz operators, oscillating symbols

We consider here a discontinuity of the second kind, the oscillating symbols. To be more precise, the following two model situation will be considered: a strong oscillation and a slow oscillation in the parabolic case. In spite of their qualitative identity, an oscillation type discontinuity, the results differ drastically.

Theorem 8.1 (Strong oscillation). *Let* $a(y) = e^{2iy}$, *then* $\mathrm{Range}\, a = S^1$ *and* $M_\infty(a) = \mathbb{D}$.

We note that for each fixed λ, the image of $\gamma_{a,\lambda}$ looks like a spiral outgoing from the point $z = 1$ and tending to $z = 0$ as x tends to 0. Moreover, when λ is growing the branches of a spiral became more closer to each other.

Theorem 8.2 (Slow oscillation). *Let* $a(y) = (2y)^i$, *then* $\mathrm{Range}\, a = S^1$ *and* $M_\infty(a) = S^1$.

Theorems 8.1 and 8.2 can be generalized to a wide class of strong and slowly oscillating symbols. For example, if $a_1(y) = (2y+1)^i$, then $M_\infty(a_1) = S^1$. For a fixed λ the image of $\gamma_{a_1,\lambda}$ is a spiral outgoing from the point $z = 1$ and tending to the limit circle with the radius $\left|\frac{\Gamma(\lambda+1+i)}{\Gamma(\lambda+1)}\right|$ and centered at origin.

We illustrate the above presenting the images of the function $\gamma_{a,\lambda}$ for two oscillating symbols

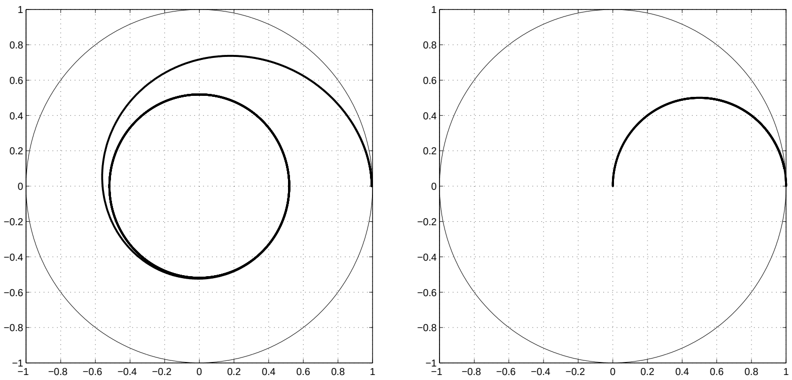
$$a_1(y) = (1+2y)^i = e^{i\ln(1+2y)} \quad \text{and} \quad a_2(y) = e^{i2y}, \quad y \in [0,\infty),$$

and for the following values of λ: 0, 10, and 1000.

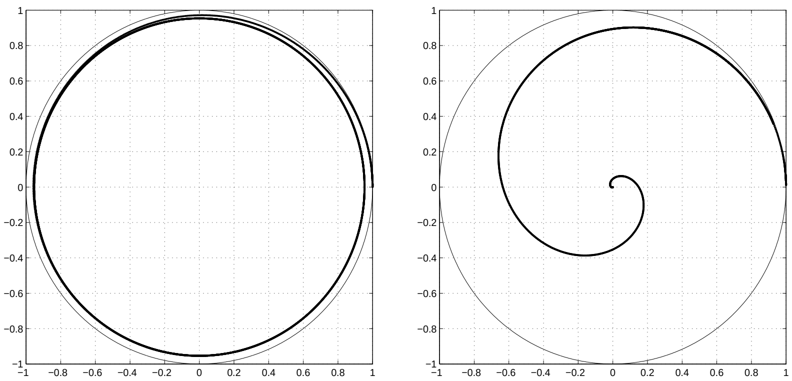
We note that from the qualitative point of view both symbols have the same properties. They are continuous at the point $y = 0$ and have the oscillation type discontinuity at infinity, both of them are of the same form

$$a_k(y) = e^{i\varphi_k(y)}, \qquad k = 1, 2,$$

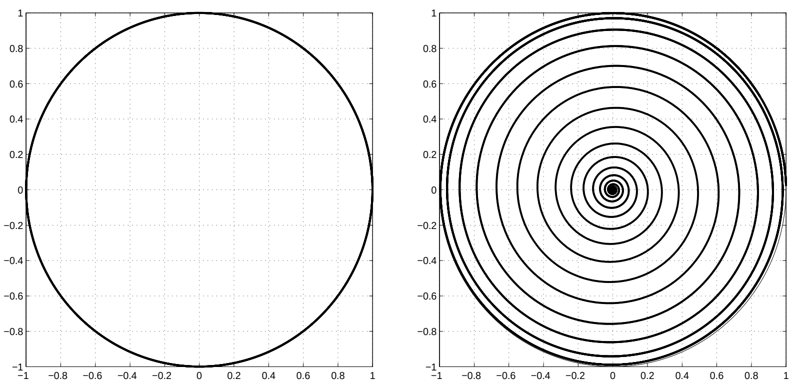
where the corresponding functions $\varphi_k(y)$ are continuous and growing in $[0,+\infty)$ with $\varphi_k(0) = 0$ and $\varphi_k(+\infty) = +\infty$. The only difference between them is on a speed of growth at infinity. And this difference leads to a drastic difference between the spectrum behaviour of the corresponding Toeplitz operators.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 0$.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 10$.



The functions $\gamma_{a_1,\lambda}(x)$ and $\gamma_{a_2,\lambda}(x)$ for $\lambda = 1000$.

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