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Integral Equations and Operator Theory

Commutative C^*-algebras of Toeplitz operators on the unit ball, II. Geometry of the level sets of symbols.

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Abstract. In the first part [16] of this work, we described the commutative C^*-algebras generated by Toeplitz operators on the unit ball $\mathbb{B}^n$ whose symbols are invariant under the action of certain Abelian groups of biholomorphisms of $\mathbb{B}^n$. Now we study the geometric properties of these symbols. This allows us to prove that the behavior observed in the case of the unit disk (see [3]) admits a natural generalization to the unit ball $\mathbb{B}^n$. Furthermore we give a classification result for commutative Toeplitz operator C^*-algebras in terms of geometric and "dynamic" properties of the level sets of generating symbols.

1. Introduction.

The commutative C^*-algebras generated by Toeplitz operators on the (weighted) Bergman spaces over the unit disk have been recently an important object of study. In [3] such C^*-algebras are completely classified in terms of the symbols of generating Toeplitz operators. It is a remarkable fact that the smoothness properties of such symbols do not play any essential role in this classification. The reason for the existence of such commutative C^*-algebras lies in the geometric and "dynamic" properties of the unit disk.

It turns out that the symbols of Toeplitz operators that generate a commutative C^*-algebra on each weighted Bergman space can be completely characterized by the geometry of their level lines. More precisely, the results in [3] show that

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the C^*-algebra generated by Toeplitz operators is commutative on each (commonly considered) weighted Bergman space if and only if there is a pencil of hyperbolic geodesics such that the symbols of the Toeplitz operators are constant on the cycles of this pencil. Here, a pencil of geodesics in the unit disk is the set of geodesics perpendicular to a cycle, i.e. a closed curve with constant geodesic curvature. All such cycles are in fact the orbits of one-parameter subgroups of isometries for the hyperbolic geometry on the unit disk. This provides the following "dynamic" restatement of the main results in [3], as long as we assume a suitable "richness" of the symbol set: *the C^*-algebra generated by Toeplitz operators is commutative on each (commonly considered) weighted Bergman space if and only if there is a maximal Abelian subgroup of the Möbius transformation group such that the symbols of the Toeplitz operators are invariant under the action of this subgroup.*

In the first part [16] of this work we started the study of the existence and behavior of the commutative C^*-algebras generated by Toeplitz operators on the unit ball $\mathbb{B}^n$. Our approach to this problem is motivated by the above discussion for the unit disk. In particular, in [16] we introduced a certain collection of Abelian subgroups of the group of biholomorphisms of $\mathbb{B}^n$. As in the case of the unit disk, it turned out that, given any such Abelian subgroup, the C^*-algebra generated by Toeplitz operators whose symbols are invariant under the action of this subgroup is commutative. This was one of the main results of [16].

In this second part of our work on the unit ball we study the geometric properties of symbol sets, generalizing the behavior observed in the case of the unit disk. At the same time, we start a program to classify the commutative C^*-algebras generated by Toeplitz operators on the unit ball. Some classification results are already given here, and in particular they show how natural is to use geometric methods in the study of such commutative C^*-algebras.

In [16] we considered $n + 2$ Abelian groups of biholomorphisms of the unit ball $\mathbb{B}^n$; these groups are listed again in Section 2 for the sake of completeness. For $n = 1$ such groups coincide, as it is readily seen, with those considered for the unit disk in [3]. Most of the groups on $\mathbb{B}^n$ are actually easier to describe by using the realization of the unit ball as the Siegel domain D_n. At the same time, it turns out that, from our geometric point of view, it is better to work with the realization given by the n-dimensional complex hyperbolic space $\mathrm{H}^n\mathbb{C}$ that is described in Section 2, and many of our results are stated for such a realization of the unit ball.

The unit ball $\mathbb{B}^n$ carries the natural Hermitian metric defined by the Bergman kernel. The associated Riemannian metric turns $\mathbb{B}^n$ into a symmetric space. For such Riemannian structure the connected component of the group of isometries is precisely the group of biholomorphisms; this is a consequence of the fact that $\mathbb{B}^n$ is a bounded symmetric domain (see [4]).

We recall in Section 2 that the group of biholomorphisms of the n-dimensional complex hyperbolic space is a Lie group whose Lie algebra is $\mathfrak{su}(n, 1)$, the skew-Hermitian complex linear transformations of the Hermitian form of signature $(n, 1)$. Using this interpretation we show in Theorem 3.6 that each of the $n + 2$ groups listed in [16] is, in fact, a maximal Abelian subgroup (MASG for short) of the group of biholomorphisms. Furthermore, Theorem 3.6 proves that, up to conjugacy, our list contains all possible MASG's of biholomorphisms of the unit ball. This already constitutes a classification result for the commutative C^*-algebras generated by Toeplitz operators on the unit ball. Indeed it shows that each MASG of biholomorphisms of the unit ball gives rise to a commutative C^*-algebra of Toeplitz operators, and moreover that all model cases of such algebras are already described in [16].

Our next goal is to study the geometric properties of the orbits of MASG's of biholomorphisms of $\mathbb{B}^n$. The importance of such goal comes from the fact that in [16] the symbol sets that generate commutative C^*-algebras of Toeplitz operators are invariant under the action of these MASG's, and thus the orbits of such MASG's correspond to the level sets of symbols. This is exactly the situation as in the case of the unit disk. Recall that, in the unit disk, we showed (see [3]) that the level sets of the symbols defining commuting algebras of Toeplitz operators exhaust all curves with constant geodesic curvature. In this work, we consider the notion of a parallel submanifold which, by Proposition 4.2, can be thought as the natural generalization of a curve with constant geodesic curvature. At the same time one has to be careful in the interpretation of such generalizations since the extrinsic geometry of curves in 2-dimensional Riemannian manifolds is very restricted. In particular, we observe that every curve in a 1-dimensional complex manifold is trivially both flat and Lagrangian; these two very important properties are not automatically satisfied by arbitrary higher dimensional submanifolds though.

With respect to this, we prove in Theorem 5.7 that every MASG H of biholomorphisms of $\mathbb{B}^n$ acts with (real) n-dimensional orbits on a connected open conull subset so that each one of the orbits is a flat parallel Lagrangian submanifold of $\mathbb{B}^n$. Moreover, Theorem 5.7 also proves that all other (lower dimensional) orbits are flat parallel Lagrangian submanifolds of a copy of some $\mathbb{B}^k$ embedded in $\mathbb{B}^n$. This shows that the geometric behavior of the level sets of the symbols considered in [3] for the unit disk extends to the orbits of the MASG's for the case of the unit ball. Furthermore, we also show in Theorem 5.9 that every flat parallel Lagrangian submanifold of $\mathbb{B}^n$ is in fact an orbit of a MASG of biholomorphisms. The last result is true even for flat parallel totally real submanifolds of $\mathbb{B}^n$.

The level sets of symbols in the unit disk are more natural to study as a whole. This turned out to be fundamental for the above mentioned classification of the commutative C^*-algebras of Toeplitz operators in the unit disk ([3]). As we already mentioned, for symbols that yield a commutative Toeplitz operator algebra, the corresponding collection of level sets is given by a flow whose normal bundle integrates to a totally geodesic flow. For a higher dimensional setup, the natural generalization of a flow is a foliation. In Theorem 6.10 we prove that every MASG H of biholomorphisms of the unit ball $\mathbb{B}^n$ defines a pair of foliations $(\mathcal{O}, \mathfrak{F})$ in a connected open conull subset, where $\mathcal{O}$ consists of H-orbits and $\mathfrak{F}$ is obtained by integrating the normal bundle to $\mathcal{O}$. Moreover, we also prove that $\mathcal{O}$ is Riemannian, $\mathfrak{F}$ is totally geodesic, and both are Lagrangian. These notions are explained in Section 6, and from the results therein it follows that the leaves of $\mathcal{O}$

are equidistant and the leaves of $\mathfrak{F}$ are made up of geodesics in $\mathbb{B}^n$. That is, such pair of foliations $(\mathcal{O}, \mathfrak{F})$ can be considered as a higher dimensional generalization of a pencil of hyperbolic geodesics (consisting of cycles and geodesics) on the unit disk.

Because of the geometric relevance of such pair of foliations we call any pair of foliations satisfying the above geometric conditions a Lagrangian frame (see Definition 6.11 for more details). We prove in Theorem 6.12 that each Lagrangian frame in $\mathbb{B}^n$ can always be obtained from a MASG of biholomorphisms. This in turn allows us to give the following classification result for families of symbols whose Toeplitz operators generate a commutative C^*-algebra. *Given any Lagrangian frame* $(\mathcal{O}, \mathfrak{F})$*, the* C^**-algebra generated by Toeplitz operators whose symbols are constant on the leaves of the foliation* $\mathcal{O}$ *is commutative in each (commonly considered) weighted Bergman space on* $\mathbb{B}^n$.

In this work we provide most of the geometric background required to understand and obtain our results. In Section 2 we define a Hopf fibration used to study the extrinsic geometry of submanifolds in the complex hyperbolic space; we introduce some machinary on the geometry of pseudo-Riemannian manifolds used further on. In Section 3, to obtain our classification of MASG's of biholomorphisms of $\mathbb{B}^n$, we state and use the classification of maximal Abelian subalgebras given in [12]. In Section 4 we describe the basics of the submanifold geometry in Riemannian manifolds; the nontrivial tools come from the classification of parallel submanifolds in the n-dimensional complex hyperbolic space as developed in [9] and [10]. The fundamentals of foliations required for this work are briefly described in Section 6.

Finally we would like to mention that we call a commutative Lie group an *Abelian* group, but we keep using the word *commutative* for operator algebras. In doing so, we try to follow the customary terminology for both Lie theory and operator theory.

2. The Hopf fibration of the complex hyperbolic space.

We denote by $\mathbb{C}^{n+1}_1$ the pseudo-Hermitian vector space with the Hermitian form given by

$$\langle z, w\rangle_1 \mapsto z^* I_{n,1} w,$$

where

$$I_{n,1} = \begin{pmatrix} I_n & 0 \\ 0 & -1 \end{pmatrix}.$$

The group of complex linear isometries of $\mathbb{C}^{n+1}_1$, denoted by $\mathrm{U}(n,1)$, consists of the $(n+1)\times(n+1)$ complex matrices A such that $A^* I_{n,1} A = I_{n,1}$. The pseudo-Hermitian space $\mathbb{C}^{n+1}_1$ has an associated pseudo-Euclidean metric

$$(z, w)_2 = \mathrm{Re}(\langle z, w\rangle_1)$$

with signature $(2n, 2)$.

The subset of $\mathbb{C}^{n+1}$ given by

$$\mathrm{H}_1^{2n+1}\mathbb{R} = \{z \in \mathbb{C}^{n+1} : \langle z, z\rangle_1 = (z, z)_2 = -1\},$$

is easily seen to be a codimension 1 smooth connected submanifold of $\mathbb{C}_1^{n+1}$.

Since the group $\mathrm{U}(n,1)$ acts by isometries on $\mathbb{C}_1^{n+1}$, it preserves $\mathrm{H}_1^{2n+1}\mathbb{R}$. Let us denote

$$\mathrm{U}_{n+1}(1) = \{tI_{n+1} : t \in \mathbb{T}\},$$

which is a Lie subgroup of $\mathrm{U}(n,1)$ isomorphic to $\mathbb{T}$.

The n-dimensional complex hyperbolic space is defined as the space

$$\mathrm{H}^n\mathbb{C} = \mathrm{U}_{n+1}(1)\backslash \mathrm{H}_1^{2n+1}\mathbb{R},$$

and the natural projection map

$$\pi : \mathrm{H}_1^{2n+1}\mathbb{R} \to \mathrm{H}^n\mathbb{C},$$

is called the Hopf fibration associated to $\mathrm{H}^n\mathbb{C}$. We will sometimes denote points in $\mathrm{H}^n\mathbb{C}$ as $\pi(z) = [z]$ for $z \in \mathrm{H}_1^{2n+1}\mathbb{R}$.

The n-dimensional complex hyperbolic space so defined will play a fundamental role in this work. We recall now some basic and well known properties of the Hopf fibration associated to $\mathrm{H}^n\mathbb{C}$.

The Hopf fibration as a smooth fiber bundle. The subgroup $\mathrm{U}_{n+1}(1)$ of $\mathrm{U}(n,1)$ acts freely on $\mathrm{H}_1^{2n+1}\mathbb{R}$ and also properly, since it is a compact group. Hence (see Theorem 1.95 of [1] and Example 10.7 of [6]), the quotient space $\mathrm{H}^n\mathbb{C}$ is a smooth manifold and the Hopf fibration $\mathrm{H}_1^{2n+1}\mathbb{R} \to \mathrm{H}^n\mathbb{C}$ defines a principal fiber bundle with structure group $\mathrm{U}_{n+1}(1) \cong \mathbb{T}$.

The Hopf fibration induces a complex structure on $\mathrm{H}^n\mathbb{C}$. From the definition of $\mathrm{H}_1^{2n+1}\mathbb{R}$ it follows that, for every $z \in \mathrm{H}_1^{2n+1}\mathbb{R}$, the tangent space $T_z\mathrm{H}_1^{2n+1}\mathbb{R}$ is the orthogonal complement (with respect to $(\cdot,\cdot)_2$) in $\mathbb{C}^{n+1}$ of the real line $\mathbb{R}z$. Let $\mathcal{H}_z(\mathrm{H}_1^{2n+1}\mathbb{R})$ be the orthogonal complement (with respect to $(\cdot,\cdot)_2$) in $T_z\mathrm{H}_1^{2n+1}\mathbb{R}$ of the real line $i\mathbb{R}z$. Then every space $\mathcal{H}_z(\mathrm{H}_1^{2n+1}\mathbb{R})$ is a complex vector subspace of $\mathbb{C}^{n+1}$ and their union defines a smooth complex vector bundle $\mathcal{H}(\mathrm{H}_1^{2n+1}\mathbb{R})$ on $\mathrm{H}_1^{2n+1}\mathbb{R}$. We will call it the horizontal bundle of the Hopf fibration, and the fibers $\mathcal{H}_z(\mathrm{H}_1^{2n+1}\mathbb{R})$ just defined will be called the horizontal spaces of the Hopf fibration.

Note that for every $z \in \mathrm{H}_1^{2n+1}\mathbb{R}$ the space $i\mathbb{R}z$ is precisely the tangent space at z of the $\mathrm{U}_{n+1}(1)$-orbit. Also, the Hopf fibration maps the horizontal spaces isomorphically (as real linear spaces) onto the tangent spaces of $\mathrm{H}^n\mathbb{C}$, thus inducing a complex structure on the latter tangent spaces. Such a structure is well defined since the $\mathrm{U}_{n+1}(1)$-action leaves invariant the horizontal bundle preserving the complex structure of its fibers. It turns out (see Example 10.7 of [6]) that $\mathrm{H}^n\mathbb{C}$ is a complex manifold with the complex structure just defined on its tangent bundle.

The Hopf fibration as a pseudo-Riemannian submersion. It will be very useful to extend the previous remarks to the category of pseudo-Riemannian manifolds. To this end we will need to consider the general notion of a pseudo-Riemannian submersion. We refer to [14] for the definition of a pseudo-Riemannian manifold.

We note that in this reference the pseudo-Riemannian manifolds are called semi-Riemannian, but besides this difference in the terminology all the basic properties listed below are as they appear in [14].

Let M and B be pseudo-Riemannian manifolds and $p : M \to B$ be a smooth submersion. We say that p is a pseudo-Riemannian submersion if the following conditions are satisfied:

- The fibers of p are nondegenerate submanifolds of M, i.e. for every $b \in B$ the tangent space to $p^{-1}(b)$ at any of its points m is a nondegenerate subspace of T_mM.
- The linear map $dp_m : (T_m p^{-1}(b))^\perp \to T_bB$ is an isometry for every $m \in M$ and $b \in B$ such that $p(m) = b$. Here $(T_m p^{-1}(b))^\perp$ is the orthogonal complement of $T_m p^{-1}(b)$ in T_mM.

As we already mentioned, the tangent space $T_z\mathrm{H}_1^{2n+1}\mathbb{R}$ to $\mathrm{H}_1^{2n+1}\mathbb{R}$ at z is the orthogonal complement with respect to $(\cdot,\cdot)_2$ of the real line $\mathbb{R}z$. Hence, the tangent space of $\mathrm{H}_1^{2n+1}\mathbb{R}$ at every point is nondegenerate. In particular, the restriction of the pseudo-Euclidean metric $(\cdot,\cdot)_2$ of $\mathbb{C}^{n+1}$ to the tangent bundle of $\mathrm{H}_1^{2n+1}\mathbb{R}$ defines a pseudo-Riemannian metric for which $\mathrm{H}_1^{2n+1}\mathbb{R}$ is a Lorentzian manifold of constant sectional curvature -1. The construction of $\mathrm{H}_1^{2n+1}\mathbb{R}$ is carried out in Chapter 4 of [14], and Proposition 29 of that chapter establishes the curvature property just mentioned. We will call $\mathrm{H}_1^{2n+1}\mathbb{R}$ the $(2n+1)$-dimensional real Lorentzian hyperbolic space.

The unitary group $\mathrm{U}(n,1)$ preserves $(\cdot,\cdot)_2$ on $\mathbb{C}^{n+1}$ and thus acts by isometries on the Lorentzian manifold $\mathrm{H}_1^{2n+1}\mathbb{R}$. In particular, the action of the group $\mathrm{U}_{n+1}(1)$ preserves the metric $(\cdot,\cdot)_2$, and so it also preserves the horizontal bundle $\mathcal{H}(\mathrm{H}_1^{2n+1}\mathbb{R})$ as well. Then, the restriction of $(\cdot,\cdot)_2$ to $\mathcal{H}(\mathrm{H}_1^{2n+1}\mathbb{R})$ defines a $\mathrm{U}_{n+1}(1)$-invariant metric which is easily seen to be positive definite. Since the Hopf fibration maps the horizontal spaces isomorphically onto the tangent spaces to $\mathrm{H}^n\mathbb{C}$, there is an induced Riemannian metric on $\mathrm{H}^n\mathbb{C}$ so that the Hopf fibration is a pseudo-Riemannian submersion. Furthermore, the complex structure on $\mathrm{H}^n\mathbb{C}$ and this Riemannian metric turn it into a Kaehler manifold with constant homolomorphic sectional curvature -4 and sectional curvature varying in the interval $[-4,-1]$. We refer to Example 10.7 of [6] where a detailed account of these constructions is presented. This yields that the n-dimensional complex hyperbolic space $\mathrm{H}^n\mathbb{C}$ is a Kaehler manifold which at the same time is the base of the pseudo-Riemannian submersion given by the Hopf fibration $\mathrm{H}_1^{2n+1}\mathbb{R} \to \mathrm{H}^n\mathbb{C}$.

O'Neill's fundamental equations of a pseudo-Riemannian submersion For a pseudo-Riemannian submersion $p : M \to B$, O'Neill introduced in [13] two tensors and a set of equations that allow to relate the geometric properties of M and B. We now recollect some definitions and facts that will be used latter on. We refer to [13] for further details. We observe that in [13] the results are stated for Riemannian submersions. However, it is well known and easy to see that for our definition of a pseudo-Riemannian submersion the results in [13] have the obvious extensions that we will state here.

Following the above notation, for a vector field X on M we denote by $\mathcal{H}(X)$ its horizontal component. Similarly, we denote by $\mathcal{V}(X)$ its vertical component. For X and Y vector fields over M we define:

$$
\begin{aligned}
T_X Y &= \mathcal{H}(\nabla_{\mathcal{V}(X)}\mathcal{V}(Y)) + \mathcal{V}(\nabla_{\mathcal{V}(X)}\mathcal{H}(Y)) \\
A_X Y &= \mathcal{V}(\nabla_{\mathcal{H}(X)}\mathcal{H}(Y)) + \mathcal{H}(\nabla_{\mathcal{H}(X)}\mathcal{V}(Y)),
\end{aligned}
$$

where ∇ is the Levi-Civita connection of M. Then, the above expressions at a point depend only on the values of X and Y at the given point. In particular, both T and A define tensors of type (1,2) over M. These are called the fundamental tensors of the pseudo-Riemannian submersion.

For u, v vector fields on either M or B we will denote:

$$Q(u,v) = \langle u,u\rangle \langle v,v\rangle - \langle u,v\rangle^2 ,$$

where $\langle \cdot,\cdot\rangle$ is the metric on either M or B, correspondingly.

The following result relates the sectional curvatures of M and B. It appears as Corollary 1 of [13]. As before, ∇ denotes the Levi-Civita connection of M. Also, for any pair of tangent vectors α, β, we denote by $P_{\alpha,\beta}$ their linear span.

Proposition 2.1 ([13]). *Let $p : M \to B$ be a pseudo-Riemannian submersion with fundamental tensors A and T. Denote by K^M, K^B and K^f the sectional curvatures of M, B and the fibers of p, respectively. If x, y are horizontal vectors and u, v are vertical vectors at some point in M such that the linear span for both x, y and u, v is a nondegenerate plane, then:*

$$
\begin{aligned}
K^M(P_{u,v}) &= K^f(P_{u,v}) - \frac{\langle T_u u, T_v v\rangle - \langle T_u v, T_u v\rangle}{Q(u,v)}, \\
K^M(P_{x,v}) \langle x,x\rangle \langle v,v\rangle &= \langle (\nabla_x T)_v v, x\rangle + \langle A_x v, A_x v\rangle - \langle T_v x, T_v x\rangle , \\
K^M(P_{x,y}) &= K^B(P_{dp(x),dp(y)}) - \frac{3\,\langle A_x y, A_x y\rangle}{Q(x,y)}.
\end{aligned}
$$

Biholomorphisms and isometries of $\mathrm{H}^n\mathbb{C}$ from its Hopf fibration. As already mentioned above, the unitary group $\mathrm{U}(n,1)$ acts by isometries on the hyperbolic space $\mathrm{H}^{2n+1}_1\mathbb{R}$. We also observe that the $\mathrm{U}_{n+1}(1)$-orbits in $\mathrm{H}^{2n+1}_1\mathbb{R}$ are precisely the fibers of the Hopf fibration. Hence, if we define the projective unitary group of signature $(n,1)$ by

$$\mathrm{PU}(n,1) = \mathrm{U}(n,1)/\mathrm{U}_{n+1}(1),$$

then there is an induced $\mathrm{PU}(n,1)$-action on the hyperbolic space $\mathrm{H}^n\mathbb{C}$. We note that this action is easily seen to be biholomorphic and isometric as a consequence of the definition of the complex and Riemannian structures on $\mathrm{H}^n\mathbb{C}$. Moreover, the action is faithful and such that the Hopf fibration is equivariant with respect to the natural quotient homomorphism of Lie groups $\psi : \mathrm{U}(n,1) \to \mathrm{PU}(n,1)$. More precisely, we have

$$\psi(A)\pi(z) = \pi(Az),$$

for every $A \in \mathrm{U}(n,1)$ and $z \in \mathrm{H}^{2n+1}_1\mathbb{R}$.

It turns out that the $\mathrm{PU}(n,1)$-action defines the full group of biholomorphisms of $\mathrm{H}^n\mathbb{C}$ and the connected component of the group of isometries. This follows, for example, by proving that $\mathrm{H}^n\mathbb{C}$ is a Riemannian symmetric space, identifying it among all such spaces and looking at the corresponding properties (see [4] and [6]). Another approach to obtain this fact is to relate the complex hyperbolic space to the complex unit ball.

From now on, for a complex manifold M we will denote by $\mathrm{Aut}(M)$ the group of its biholomorphisms. Hence, we just observed that $\mathrm{Aut}(\mathrm{H}^n\mathbb{C}) = \mathrm{PU}(n,1)$.

For ease of reference, we resume the above remarks in the following statement.

Theorem 2.2. *The space* $\mathrm{H}^{2n+1}_1\mathbb{R}$ *is a Lorentz manifold and* $\mathrm{H}^n\mathbb{C}$ *is a Kaehler Riemannian manifold such that the Hopf fibration*

$$\pi : \mathrm{H}^{2n+1}_1\mathbb{R} \to \mathrm{H}^n\mathbb{C}$$

is a pseudo-Riemannian submersion. Furthermore, if $\psi : \mathrm{U}(n,1) \to \mathrm{PU}(n,1)$ *denotes the quotient homomorphism, then* π *is* ψ*-equivariant, i.e.,*

$$\pi(Az) = \psi(A)(\pi(z)),$$

for every $A \in \mathrm{U}(n,1)$ *and* $z \in \mathrm{H}^{2n+1}_1\mathbb{R}$*. Also, we have that* $\mathrm{Aut}(\mathrm{H}^n\mathbb{C}) = \mathrm{PU}(n,1)$.

The unit ball $\mathbb{B}^n$ as a realization of $\mathrm{H}^n\mathbb{C}$. Let us denote by $\mathbb{B}^n$ the unit ball in the complex vector space $\mathbb{C}^n$, in other words we have

$$\mathbb{B}^n = \{z = (z_1, ..., z_n) \in \mathbb{C}^n : \ |z|^2 = |z_1|^2 + ... + |z_n|^2 < 1\}.$$

Then it is easily seen that the map:

$$\begin{aligned} \varphi_1 : \mathrm{H}^n\mathbb{C} &\to \mathbb{B}^n \\ [z] &\mapsto \frac{z'}{z_{n+1}}, \end{aligned}$$

defines a biholomorphism that realizes the n-dimensional complex hyperbolic space as a bounded domain in $\mathbb{C}^n$. This biholomorphism allows us to attach to $\mathbb{B}^n$ the above structures built for $\mathrm{H}^n\mathbb{C}$. In particular, from Section 3.1 of [2] it follows that the Hermitian metric on $\mathbb{B}^n$ is given by

$$ds^2_{\mathbb{B}^n} = \frac{1}{1 - \sum_{k=1}^n |z_k|^2} \left(\sum_{k=1}^n dz^k \otimes d\overline{z}^k + \sum_{k,l=1}^n \frac{\overline{z}_k z_l \, dz^k \otimes d\overline{z}^l}{1 - \sum_{k=1}^n |z_k|^2} \right)$$

which is normalized so that it has constant holomorphic sectional curvature -4.

In the rest of this work, we will use the notation $z = (z', z_k)$, where $z' = (z_1, ..., z_{k-1}) \in \mathbb{C}^{k-1}$ and $z_k \in \mathbb{C}$.

The Siegel domain D_n as a realization of $\mathrm{H}^n\mathbb{C}$. The above constructions do not depend on the matrix $I_{n,1}$ defining the Hermitian form $\langle \cdot, \cdot \rangle_1$, and we can replace it with any other matrix that defines a Hermitian form with the same signature $(n, 1)$.

Let B be a complex $(n+1)\times(n+1)$ Hermitian matrix of signature $(n,1)$, i.e. such that its associated Hermitian form in $\mathbb{C}^{n+1}$

$$(z,w)\mapsto z^*Bw,$$

is a nondegenerate Hermitian form of signature $(n,1)$ on $\mathbb{C}^{n+1}$. We will denote by the same symbol B such Hermitian form. We also denote by

$$\mathrm{U}(\mathbb{C}^{n+1},B)=\{A\in\mathbb{C}^{(n+1)\times(n+1)}:A^*BA=B\},$$

the unitary group associated to B and observe that $\mathrm{U}_{n+1}(1)$ is also a subgroup of $\mathrm{U}(\mathbb{C}^{n+1},B)$. Then the projective unitary group associated to B is given by

$$\mathrm{PU}(\mathbb{C}^{n+1},B)=\mathrm{U}(\mathbb{C}^{n+1},B)/\mathrm{U}_{n+1}(1).$$

We can also define hyperbolic spaces as before by

$$\begin{aligned}\mathrm{H}^{2n+1}_B\mathbb{R}&=\{z\in\mathbb{C}^{n+1}:B(z,z)=-1\},\\ \mathrm{H}^n_B\mathbb{C}&=\mathrm{U}_{n+1}(1)\backslash\mathrm{H}^{2n+1}_B\mathbb{R},\end{aligned}$$

with the quotient map as the corresponding Hopf fibration. Then, it is straightforward to check that we have all of the properties stated above for this new setup. Moreover, since B has signature $(n,1)$, there exists a nondegenerate $(n+1)\times(n+1)$ complex matrix A for which $A^*BA=I_{n,1}$. Such an A considered as a complex linear map $\mathbb{C}^{n+1}_1\to(\mathbb{C}^{n+1},B)$ defines an isometry of pseudo-Hermitian spaces. Also, this isometry of pseudo-Hermitian spaces yields the isomorphism of Lie groups

$$\begin{aligned}\mathrm{U}(n,1)&\to\mathrm{U}(\mathbb{C}^{n+1},B)\\ T&\mapsto ATA^{-1},\end{aligned}$$

and a corresponding isomorphism of the projective unitary groups.

Then, the linear map A defines an isometry $\mathrm{H}^{2n+1}_1\mathbb{R}\to\mathrm{H}^{2n+1}_B\mathbb{R}$ of Lorentzian manifolds that induces a corresponding biholomorphic isometry $\mathrm{H}^n\mathbb{C}\to\mathrm{H}^n_B\mathbb{C}$. Also, such equivalences are equivariant with respect to the above isomorphisms of the corresponding unitary and projective unitary groups. The corresponding Hopf fibrations are equivalent as well.

Our main reason to discuss this general situation is to consider the Siegel domain

$$D_n=\{z=(z',z_n)\in\mathbb{C}^{n-1}\times\mathbb{C}:\mathrm{Im}(z_n)-|z'|^2>0\},$$

which is useful to study the biholomorphisms of the unit ball.

Consider the matrix

$$K_n=\begin{pmatrix}2I_{n-1}&0&0\\0&0&-i\\0&i&0\end{pmatrix},$$

which is easily seen to have signature $(n,1)$. Then the map

$$\begin{aligned}\varphi_2 : \mathrm{H}^n_{K_n}\mathbb{C} &\to D_n \\ [z] &\mapsto \frac{z'}{z_{n+1}},\end{aligned}$$

is a biholomorphism and thus realizes the n-dimensional complex hyperbolic space $\mathrm{H}^n_{K_n}\mathbb{C}$ as the Siegel domain D_n. The corresponding Hermitian metric of D_n, normalized to constant holomorphic sectional curvature -4, is given by

$$\begin{aligned} ds^2_{D_n} = \frac{1}{\mathrm{Im}(z_n) - |z'|^2} &\left(\frac{dz_n \otimes d\overline{z}_n}{4(\mathrm{Im}(z_n) - |z'|^2)} + \sum_{k=1}^{n-1} dz_k \otimes d\overline{z}_k \right. \\ &\left. + \frac{1}{2i} \sum_{k=1}^{n-1} \frac{\overline{z}_k dz_k \otimes d\overline{z}_n - z_k dz_n \otimes d\overline{z}_k}{\mathrm{Im}(z_n) - |z'|^2} + \sum_{k,l=1}^{n-1} \frac{\overline{z}_k z_l dz_k \otimes d\overline{z}_l}{\mathrm{Im}(z_n) - |z'|^2} \right). \end{aligned}$$

Biholomorphisms of $\mathbb{B}^n$ and D_n. From the previous remarks and equivalences we now write down explicitly the actions of the projective unitary groups on the unit ball and on the Siegel domain. In the rest of this work, for a matrix A in a unitary group we will denote by $[A]$ its class in the corresponding projective unitary group. Similarly, if we represent a unitary matrix by some array (A_{ij}), then its class in the projective unitary group will be represented by $[A_{ij}]$.

Proposition 2.3. *The group* $\mathrm{Aut}(\mathbb{B}^n)$ *of biholomorphisms of* $\mathbb{B}^n$ *is realized by the action*

$$\begin{gathered}\mathrm{PU}(n,1) \times \mathbb{B}^n \to \mathbb{B}^n \\ \begin{bmatrix} A & b \\ c & d \end{bmatrix} \cdot z = \frac{Az+b}{c\cdot z + d},\end{gathered}$$

where A *is an* $n\times n$ *matrix,* $d \in \mathbb{C}$ *and the other matrix entries have corresponding sizes.*

Proposition 2.4. *The group* $\mathrm{Aut}(D_n)$ *of biholomorphisms of* D_n *is realized by the action*

$$\begin{gathered}\mathrm{PU}(\mathbb{C}^{n+1}, K_n) \times D_n \to D_n \\ \begin{bmatrix} A & \alpha & \beta \\ \gamma & a & b \\ \delta & c & d \end{bmatrix} \cdot (z', z_n) = \frac{(Az' + \alpha\cdot z_n + \beta, \gamma\cdot z' + a z_n + b)}{\delta \cdot z' + c z_n + d},\end{gathered}$$

where A *is an* $(n-1)\times(n-1)$ *matrix,* $a, d \in \mathbb{C}$ *and the other matrix entries have corresponding sizes.*

The realizations of $\mathrm{H}^n\mathbb{C}$ as the unit ball $\mathbb{B}^n$ and as the Siegel domain D_n, together with the above propositions on their biholomorphisms, implies that any result stated for either of the three of them provides immediately a corresponding result for the other two. Thus we will always assume that we do have a desired property for all three realizations of the n-dimensional complex hyperbolic space whenever we have proved it for just one of them.

As discussed in the Introduction, we will be mostly interested in actions of Abelian subgroups of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. This motivates the following definition.

Definition 2.5. A subset M of $\mathrm{H}^n\mathbb{C}$ will be called an Abelian orbit if it is an orbit of a connected Abelian subgroup of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$.

We list now the following $n+2$ Abelian subgroups of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ realized as biholomorphic actions on either $\mathbb{B}^n$ or D_n. They all define Abelian orbits that played a fundamental role in [16].

The quasi-elliptic group of biholomorphisms of the unit ball $\mathbb{B}^n$ is isomorphic to $\mathbb{T}^n$ with the group action

$$t: \ z = (z_1, ..., z_n) \in \mathbb{B}^n \longmapsto tz = (t_1 z_1, ..., t_n z_n) \in \mathbb{B}^n,$$

for each $t = (t_1, ..., t_n) \in \mathbb{T}^n$. We will denote by $\mathfrak{E}(n)$ the group of automorphisms for this action.

The quasi-parabolic group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}$ with the group action

$$(t, h): \ (z', z_n) \in D_n \longmapsto (tz', z_n + h) \in D_n,$$

for each $(t, h) \in \mathbb{T}^{n-1} \times \mathbb{R}$. We will denote by $\mathfrak{P}(n)$ the group of automorphisms given by this action.

The quasi-hyperbolic group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}_+$ with the group action

$$(t, r): \ (z', z_n) \in D_n \longmapsto (r^{1/2} tz', r z_n) \in D_n,$$

for each $(t, r) \in \mathbb{T}^{n-1} \times \mathbb{R}_+$. We will denote by $\mathfrak{H}(n)$ the group of automorphisms given by this action.

The nilpotent group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{R}^{n-1} \times \mathbb{R}$ with the group action

$$(b, h): \ (z', z_n) \in D_n \mapsto (z' + b, z_n + h + 2iz' \cdot b + i|b|^2) \in D_n,$$

for each $(b, h) \in \mathbb{R}^{n-1} \times \mathbb{R}$. We will denote by $\mathfrak{N}(n)$ the group of automorphisms given by this action.

The quasi-nilpotent group of biholomorphisms of the Siegel domain D_n is isomorphic to $\mathbb{T}^k \times \mathbb{R}^{n-k-1} \times \mathbb{R}$, $0 < k < n-1$, with the group action

$$(t, b, h): \ (z', z'', z_n) \in D_n \longmapsto (tz', z'' + b, z_n + h + 2iz'' \cdot b + i|b|^2) \in D_n,$$

where we have decomposed the vectors involved such that $z' \in \mathbb{C}^k, z'' \in \mathbb{C}^{n-k-1}$. We will denote by $\mathfrak{N}(n, k)$ the group of automorphisms given by this action. To be more specific we will call $\mathfrak{N}(n, k)$ the quasi-nilpotent group of index (n, k). Observe that our restrictions on k are given to avoid repetition with the quasi-parabolic and nilpotent types, since the above action reduces to the former for $k = n-1$ and to the latter for $k = 0$.

3. Classification of the Maximal Abelian subgroups of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$.

In this section we show that the Abelian subgroups of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ defined at the end of Section 2 exhaust all possibilities. For this, we will use the following definition of maximality.

Definition 3.1. If G is a Lie group, then a maximal connected Abelian subgroup H of G is a Lie subgroup of G that satisfies

- H is connected Abelian,
- if H_1 is a connected Abelian subgroup of G and $H_1 \supset H$, then $H_1 = H$.

For brevity, we will say that such subgroup is a MASG of G.

Corresponding to this and Definition 2.5, an orbit of a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ will be called a MASG orbit in $\mathrm{H}^n\mathbb{C}$.

To take into account obvious identifications, i.e. $\mathrm{H}^n\mathbb{C}$ realized either as $\mathbb{B}^n$ or D_n, we will consider the following equivalence relation.

Definition 3.2. Let M and M' be connected complex manifolds. If H and H' are subgroups of $\mathrm{Aut}(M)$ and $\mathrm{Aut}(M')$, respectively, we will say that (H, M) and (H', M) are analytically equivalent if there is a biholomorphism $\varphi : M' \to M$ such that $H' = \varphi^{-1} H \varphi$.

Our aim now is to determine the equivalence classes of pairs $(H, \mathrm{H}^n\mathbb{C})$ for a connected Abelian subgroup H. As it is often done in Lie group theory, we will solve this problem by considering the corresponding problem for Lie algebras.

Definition 3.3. Let $\mathfrak{g}$ be a Lie algebra. A Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is called a maximal Abelian subalgebra, or MASA for short, when

- $\mathfrak{h}$ is Abelian,
- if $\mathfrak{h}_1$ is an Abelian subalgebra of $\mathfrak{g}$ and $\mathfrak{h}_1 \supset \mathfrak{h}$, then $\mathfrak{h}_1 = \mathfrak{h}$.

We say that a Lie algebra $\mathfrak{g}$ is linear if it is a real Lie subalgebra of $\mathfrak{gl}(n, \mathbb{C})$, for some n. In such case, to understand MASA's of $\mathfrak{g}$ it is enough to describe them up to changes of coordinates. More precisely, if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are linear Lie algebras in $\mathfrak{gl}(n, \mathbb{C})$, with Abelian subalgebras $\mathfrak{h}_1$, $\mathfrak{h}_2$, respectively, we say that the pairs $(\mathfrak{h}_1, \mathfrak{g}_1)$ and $(\mathfrak{h}_2, \mathfrak{g}_2)$ are conjugate if there exists $A \in \mathrm{GL}(n, \mathbb{C})$ such that $\mathfrak{g}_2 = A^{-1}\mathfrak{g}_1 A$ and $\mathfrak{h}_2 = A^{-1}\mathfrak{h}_1 A$.

We denote by $\mathfrak{su}(n, 1)$ the Lie algebra of traceless derivations of the Hermitian form on $\mathbb{C}^{n+1}_1$, in other words, we have

$$\mathfrak{su}(n, 1) = \{A \in \mathfrak{gl}(n + 1, \mathbb{C}) : A^* I_{n,1} + I_{n,1} A = 0,\ \mathrm{tr}(A) = 0\}.$$

More generally, if B is a complex $(n + 1) \times (n + 1)$ Hermitian matrix of signature $(n, 1)$, then we denote

$$\mathfrak{su}(\mathbb{C}^{n+1}, B) = \{A \in \mathfrak{gl}(n + 1, \mathbb{C}) : A^* B + BA = 0,\ \mathrm{tr}(A) = 0\},$$

which is the Lie algebra of traceless derivations of the Hermitian form associated to B given by the assignment

$$(z, w) \mapsto z^* Bw.$$

We recall that $\mathfrak{su}(\mathbb{C}^{n+1}, B)$ is the Lie algebra of both of the Lie groups $\mathrm{SU}(\mathbb{C}^{n+1}, B)$ and $\mathrm{PU}(\mathbb{C}^{n+1}, B)$. In particular, $\mathfrak{su}(n,1)$ is the Lie algebra of the projective unitary group $\mathrm{PU}(n,1)$.

It turns out that some MASA's of $\mathfrak{su}(n,1)$ are easier to describe for various other choices of B different from $I_{n,1}$. Also, for this setup, we consider a stronger form of conjugacy.

Definition 3.4. Let B and B' be Hermitian forms on $\mathbb{C}^{n+1}$ of signature $(n,1)$ and $\mathfrak{h}$, $\mathfrak{h}'$ Abelian subalgebras of $\mathfrak{su}(\mathbb{C}^{n+1}, B)$, $\mathfrak{su}(\mathbb{C}^{n+1}, B')$, respectively. We say that $(\mathfrak{h}, \mathfrak{su}(\mathbb{C}^{n+1}, B))$ and $(\mathfrak{h}', \mathfrak{su}(\mathbb{C}^{n+1}, B'))$ are unitarily equivalent if there exists some $A \in \mathrm{GL}(n+1, \mathbb{C})$ such that:

$$B' = A^* B A$$
$$\mathfrak{su}(\mathbb{C}^{n+1}, B') = A^{-1} \mathfrak{su}(\mathbb{C}^{n+1}, B) A$$
$$\mathfrak{h}' = A^{-1} \mathfrak{h} A.$$

A classification of MASA's of $\mathfrak{su}(n,1)$ up to unitary equivalence is given in [12]. The next statement is essentially Theorem 5.1 from [12] except for a trivial change of coordinates that we explain for the sake of completeness. In what follows besides the matrix K_n defined before, we will consider the matrix

$$L_n = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

which is easily seen to have signature $(n,1)$.

Theorem 3.5 ([12]). *For every k, let $D(k)$ be the space of $k \times k$ diagonal matrices with imaginary entries. Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{su}(n,1)$. Then $\mathfrak{h}$ is a MASA of $\mathfrak{su}(n,1)$ if and only if the pair $(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to one of the following pairs. Furthermore, no two of these pairs are unitarily equivalent.*

1. $(\mathfrak{e}(n,1), \mathfrak{su}(n,1))$*, where* $\mathfrak{e}(n,1) = D(n+1) \cap \mathfrak{su}(n,1)$ *is the Lie subalgebra of diagonal matrices in* $\mathfrak{su}(n,1)$*.*
2. $(\mathfrak{h}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$*, for the Lie algebra* $\mathfrak{h}(n)$ *of matrices of the form:*
$$\begin{pmatrix} D & 0 & 0 \\ 0 & z & 0 \\ 0 & 0 & -\overline{z} \end{pmatrix},$$
where $z \in \mathbb{C}, D \in D(n-1)$ *and* $\mathrm{tr}(D) + 2i\mathrm{Im}(z) = 0$*.*
3. $(\mathfrak{p}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$*, for the Lie algebra* $\mathfrak{p}(n)$ *of matrices of the form:*
$$\begin{pmatrix} D & 0 & 0 \\ 0 & iy & a \\ 0 & 0 & iy \end{pmatrix},$$
where $a, y \in \mathbb{R}, D \in D(k)$ *and* $\mathrm{tr}(D) + 2iy = 0$*.*

4. $(\mathfrak{n}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$, *for the Lie algebra* $\mathfrak{n}(n)$ *of matrices of the form:*
$$\begin{pmatrix} 0 & 0 & b^t \\ 2ib & 0 & a \\ 0 & 0 & 0 \end{pmatrix},$$
where $a \in \mathbb{R}, b \in \mathbb{R}^{n-1}$.
5. $(\mathfrak{n}(n,k), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$, *for some* k *such that* $1 \leq k \leq n-2$, *and for the Lie algebra* $\mathfrak{n}(n,k)$ *of matrices of the form:*
$$\begin{pmatrix} D & 0 & 0 & 0 \\ 0 & iyI_{n-k-1} & 0 & b^t \\ 0 & 2ib & iy & a \\ 0 & 0 & 0 & iy \end{pmatrix},$$
where $a, y \in \mathbb{R}, b \in \mathbb{R}^{n-1}, D \in D(k)$ *and* $\mathrm{tr}(D) + iy(n-k+1) = 0$.

Proof. From the discussion found in Theorem 5.1 of [12] it follows that, up to unitary equivalence, all MASA's of $\mathfrak{su}(n,1)$ are given by $\mathfrak{e}(n,1)$, the Lie subalgebra $\mathfrak{g}'$ of $\mathfrak{su}(\mathbb{C}^{n+1}, L_n)$ defined as:
$$\mathfrak{g}' = \left\{ \begin{pmatrix} z & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & -\overline{z} \end{pmatrix} : z \in \mathbb{C}, D \in D(n-1), \mathrm{tr}(D) + 2i\mathrm{Im}(z) = 0 \right\},$$
and the Lie subalgebras $\mathfrak{g}_k$ of $\mathfrak{su}(\mathbb{C}^{n+1}, L_n)$ $(0 \leq k \leq n-1)$ consisting of all matrices of the form
$$\begin{pmatrix} iy & 0 & b & ia \\ 0 & D & 0 & 0 \\ 0 & 0 & iyI_{n-k-1} & -b^t \\ 0 & 0 & 0 & iy \end{pmatrix},$$
where $a, y \in \mathbb{R}, b \in \mathbb{R}^{n-k-1}, D \in D(k)$ and $iy(n-k+1) + \mathrm{tr}(D) = 0$.
Let us consider the matrix
$$A_k = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & I_k & 0 & 0 \\ 0 & 0 & iI_{n-k-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
Then it is immediate to check that $A_k^* L_n A_k = L_n$. From this it follows that, for every k as above, the Lie algebra $\mathfrak{g}_k' = A_k^{-1} \mathfrak{g}_k A_k$ is a Lie subalgebra of $\mathfrak{su}(\mathbb{C}^{n+1}, L_n)$ in the same class of unitary equivalence as that of $\mathfrak{g}_k$. A simple computation shows that $\mathfrak{g}_k'$ is the set of matrices of the form
$$\begin{pmatrix} iy & 0 & ib & ia \\ 0 & D & 0 & 0 \\ 0 & 0 & iyI_{n-k-1} & ib^t \\ 0 & 0 & 0 & iy \end{pmatrix},$$
where $a, y \in \mathbb{R}, b \in \mathbb{R}^{n-k-1}, D \in D(k)$ and $iy(n-k+1) + \mathrm{tr}(D) = 0$.

Let us now consider the matrix

$$A = \begin{pmatrix} 0 & i & 0 \\ -\sqrt{2}iI_{n-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

it is a simple matter to check that $K_n = A^* L_n A$. Then the result follows from the following identities obtained by explicit computation:

$$\begin{aligned} \mathfrak{h}(n) &= A^{-1}\mathfrak{g}' A \\ \mathfrak{p}(n) &= A^{-1}\mathfrak{g}'_{n-1} A \\ \mathfrak{n}(n) &= A^{-1}\mathfrak{g}'_0 A \\ \mathfrak{n}(n,k) &= A^{-1}\mathfrak{g}'_k A, \end{aligned}$$

for $1 \leq k \leq n-2$. Here we use that

$$A^{-1} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}}iI_{n-1} & 0 \\ -i & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

□

Based on the above, the next result provides a complete description, up to analytic equivalence, of the MASG's in $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$.

Theorem 3.6. *Let H be a connected subgroup of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. Then H is a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ if and only if $(H, \mathrm{H}^n\mathbb{C})$ is analytically equivalent to one of the following pairs:*

1. $(\mathfrak{E}(n), \mathbb{B}^n)$ *for* $n \geq 1$,
2. $(\mathfrak{H}(n), D_n)$ *for* $n \geq 1$,
3. $(\mathfrak{P}(n), D_n)$ *for* $n \geq 1$,
4. $(\mathfrak{N}(n), D_n)$ *for* $n \geq 2$
5. $(\mathfrak{N}(n,k), D_n)$ *for* $1 \leq k \leq n-2$.

In particular, any subgroup H from the above list is a MASG in either $\mathrm{Aut}(\mathbb{B}^n)$ or $\mathrm{Aut}(D_n)$, according to which of these two contains H. Furthermore, no two of these pairs are analytically equivalent.

Proof. Let H be a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. By Theorem 2.2, the group of biholomorphisms is realized by the projective unitary group $\mathrm{PU}(n,1)$ whose Lie algebra is $\mathfrak{su}(n,1)$. Hence, if $\mathfrak{h}$ is the Lie algebra of H, then it can be considered as a Lie subalgebra of $\mathfrak{su}(n,1)$. Moreover, by the correspondence between Lie subalgebras and connected Lie subgroups, it follows that $\mathfrak{h}$ is a MASA in $\mathfrak{su}(n,1)$. By Theorem 3.5 the pair $(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to one of the pairs in its statement. We now prove that $(H, \mathrm{Aut}(\mathrm{H}^n\mathbb{C}))$ is analytically equivalent to groups of biholomorphisms of either $\mathbb{B}^n$ or D_n considering the following cases.

$(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to $(\mathfrak{e}(n,1), \mathfrak{su}(n,1))$. Let $A \in \mathrm{U}(n,1)$ be such that $A^* I_{n,1} A = I_{n,1}$ and $\mathfrak{h} = A^{-1}\mathfrak{e}(n,1)A$. In particular, $[A] \in \mathrm{PU}(n,1)$. Since $\mathfrak{e}(n,1)$ is the Lie algebra of the subgroup $E(n,1)$ of $\mathrm{PU}(n,1)$ with diagonal matrices as representatives (in the $\mathrm{U}_{n+1}(1)$ class), we have that $[A]^{-1}E(n,1)[A] = H$. Clearly, the action of $E(n,1)$ on $\mathbb{B}^n$ defines the MASG $\mathfrak{E}(n)$ of $\mathrm{Aut}(\mathbb{B}^n)$, and so $[A]$ is a biholomorphism of $\mathbb{B}^n$ with respect to which $(H, \mathrm{Aut}(\mathbb{B}^n))$ and $(\mathfrak{E}(n), \mathbb{B}^n)$ are analytically equivalent.

$(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to $(\mathfrak{h}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$. First we replace $(H, \mathbb{B}^n)$ by an analytically equivalent pair (H_1, D_n). Thus the pair of Lie algebras $(\mathrm{Lie}(H_1), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$ is unitarily equivalent to $(\mathfrak{h}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$ as well. Hence, we can choose $A \in \mathrm{U}(\mathbb{C}^{n+1}, K_n)$ such that $A^*K_nA = K_n$ and $\mathrm{Lie}(H_1) = A^{-1}\mathfrak{h}(n)A$. From Proposition 2.4 it follows that $[A] \in \mathrm{Aut}(D_n)$. By exponentiating matrices, it is easy to see that the connected Lie subgroup of $\mathrm{PU}(\mathbb{C}^{n+1}, K_n)$ with Lie algebra $\mathfrak{h}(n)$ consists of those classes (modulo $\mathrm{U}_{n+1}(1)$) whose representative matrices are of the form

$$t \begin{pmatrix} D & 0 & 0 \\ 0 & r & 0 \\ 0 & 0 & r^{-1} \end{pmatrix},$$

where $r \in \mathbb{R}_+$, $t \in \mathbb{T}$ and D is an $(n-1) \times (n-1)$ diagonal matrix with diagonal entries in $\mathbb{T}$ such that $t^{n+1}\det(D) = 1$. According to Proposition 2.4, this last group of matrices acts on D_n realizing the group $\mathfrak{H}(n)$. We have that

$$\begin{aligned} \mathrm{Aut}(D_n) &= [A]^{-1}\mathrm{Aut}(D_n)[A] \\ H_1 &= [A]^{-1}\mathfrak{H}(n)[A], \end{aligned}$$

which implies that $(H_1, \mathrm{Aut}(D_n))$ is analytically equivalent to $(\mathfrak{H}(n), D_n)$. Hence, $(H, \mathrm{Aut}(\mathbb{B}^n))$ is analytically equivalent to $(\mathfrak{H}(n), D_n)$ as well.

$(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to $(\mathfrak{p}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$. As above, we first replace $(H, \mathbb{B}^n)$ by an analytically equivalent pair (H_1, D_n). Hence, the pair $(\mathrm{Lie}(H_1), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$ is unitarily equivalent to $(\mathfrak{p}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$, and we can choose $A \in \mathrm{U}(\mathbb{C}^{n+1}, K_n)$ such that $A^*K_nA = K_n$ and $\mathrm{Lie}(H_1) = A^{-1}\mathfrak{p}(n,k)A$. We have again that $[A] \in \mathrm{Aut}(D_n)$. In this case, the connected Lie subgroup of $\mathrm{PU}(\mathbb{C}^{n+1}, K_n)$ with Lie algebra $\mathfrak{p}(n)$ is the set of classes with matrix representatives of the form

$$t \begin{pmatrix} D & 0 & 0 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix},$$

where $a \in \mathbb{R}$, $t \in \mathbb{T}$ and D is a $(n-1) \times (n-1)$ diagonal matrix with entries in $\mathbb{T}$ such that $t^{n+1}\det(D) = 1$. Now, according to Proposition 2.4, this last group of matrices acts on D_n realizing the group $\mathfrak{P}(n)$. And so it follows that

$$\begin{aligned} \mathrm{Aut}(D_n) &= [A]^{-1}\mathrm{Aut}(D_n)[A] \\ H_1 &= [A]^{-1}\mathfrak{P}(n)[A]. \end{aligned}$$

As before, this implies that $(H, \mathrm{Aut}(\mathbb{B}^n))$ is analytically equivalent to the pair $(\mathfrak{P}(n), D_n)$.

$(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to $(\mathfrak{n}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$. Again, we first replace $(H, \mathbb{B}^n)$ by an analytically equivalent pair (H_1, D_n), so that the pair of Lie

algebras
$(\mathrm{Lie}(H_1), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$ is unitarily equivalent to $(\mathfrak{n}(n), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$. Then we choose $A \in \mathrm{U}(\mathbb{C}^{n+1}, K_n)$ such that $A^* K_n A = K_n$ and $\mathrm{Lie}(H_1) = A^{-1}\mathfrak{n}(n)A$. We also have $[A] \in \mathrm{Aut}(D_n)$. In this case, the connected Lie subgroup of $\mathrm{PU}(\mathbb{C}^{n+1}, K_n)$ with Lie algebra $\mathfrak{n}(n)$ is the set of classes of matrices of the form

$$\begin{pmatrix} I_{n-1} & 0 & b^t \\ 2ib & 1 & a + i|b|^2 \\ 0 & 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{R}^{n-1}$ and $a \in \mathbb{R}$. Using Proposition 2.4, this last group of matrices acts on D_n realizing the group $\mathfrak{N}(n)$. We now obtain the relations:

$$\begin{aligned} \mathrm{Aut}(D_n) &= [A]^{-1}\mathrm{Aut}(D_n)[A] \\ H_1 &= [A]^{-1}\mathfrak{N}(n)[A]. \end{aligned}$$

This implies that $(H, \mathrm{Aut}(\mathbb{B}^n))$ is analytically equivalent to $(\mathfrak{N}(n), D_n)$.

$(\mathfrak{h}, \mathfrak{su}(n,1))$ is unitarily equivalent to $(\mathfrak{n}(n,k), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$. For this case we are considering $1 \leq k \leq n-2$. Then, we replace $(H, \mathbb{B}^n)$ by an analytically equivalent pair (H_1, D_n), and obtain a pair of Lie algebras $(\mathrm{Lie}(H_1), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$ that is unitarily equivalent to $(\mathfrak{n}(n,k), \mathfrak{su}(\mathbb{C}^{n+1}, K_n))$. Then we choose $A \in \mathrm{U}(\mathbb{C}^{n+1}, K_n)$ such that $A^* K_n A = K_n$ and $\mathrm{Lie}(H_1) = A^{-1}\mathfrak{n}(n,k)A$. In particular, $[A] \in \mathrm{Aut}(D_n)$. In this case, the connected Lie subgroup of $\mathrm{PU}(\mathbb{C}^{n+1}, K_n)$ with Lie algebra $\mathfrak{n}(n,k)$ is the set of classes with representatives of the form

$$t \begin{pmatrix} D & 0 & 0 & 0 \\ 0 & I_{n-k-1} & 0 & b^t \\ 0 & 2ib & 1 & a + i|b|^2 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $b \in \mathbb{R}^{n-k-1}$, $a \in \mathbb{R}$, $t \in \mathbb{T}$ and D is a $k \times k$ diagonal matrix with entries in $\mathbb{T}$ such that $t^{n+1}\det(D) = 1$. Using Proposition 2.4 once more, this last group of matrices acts on D_n realizing the group $\mathfrak{N}(n,k)$. And so we obtain the relations

$$\begin{aligned} \mathrm{Aut}(D_n) &= [A]^{-1}\mathrm{Aut}(D_n)[A] \\ H_1 &= [A]^{-1}\mathfrak{N}(n,k)[A]. \end{aligned}$$

As before, this implies that $(H, \mathrm{Aut}(\mathbb{B}^n))$ is analytically equivalent to the pair $(\mathfrak{N}(n,k), D_n)$.

For the converse, note that the above arguments have shown that the pairs listed in the statement are given by Lie subgroups whose Lie subalgebras are conjugate to MASA's of $\mathfrak{su}(n,1)$. Also, we recall that for connected Lie subgroups H_1 and H_2 of any Lie group we have $H_1 \subset H_2$ if and only if $\mathrm{Lie}(H_1) \subset \mathrm{Lie}(H_2)$. From these remarks it follows that the Lie subgroups coming from the pairs listed in the statement are indeed MASG's.

To prove the last claim, we observe that if $(H, \mathrm{Aut}(\mathrm{H}^n\mathbb{C}))$ is analytically equivalent to $(H', \mathrm{Aut}(M'))$, then H and H' are isomorphic. From this it follows that, for the above restrictions on the values of n and k given above, none of the pairs in the statement can be analytically equivalent except for $(\mathfrak{H}(n), D_n)$ and $(\mathfrak{P}(n), D_n)$, which correspond to actions of $\mathbb{T}^{n-1} \times \mathbb{R}_+$ and $\mathbb{T}^{n-1} \times \mathbb{R}$, respectively. To see that these two pairs are not analytically equivalent it is enough to observe that $\mathfrak{H}(n)$ has exactly two fixed points in the boundary of D_n, whereas $\mathfrak{P}(n)$ has just one fixed point in the boundary of D_n. □

4. Geometry of the submanifolds of $\mathrm{H}^n\mathbb{C}$.

We start with the basic notions and notations of the extrinsic geometry of a submanifold in a pseudo-Riemannian manifold. We refer to [14] and [6] for further details.

Let $\overline{M}$ be a pseudo-Riemannian manifold and let M be a pseudo-Riemannian submanifold of $\overline{M}$. In other words, M is a submanifold so that T_xM is a nondegenerate subspace of $T_x\overline{M}$ for every $x \in M$. In particular, the pseudo-Riemannian metric on $\overline{M}$ when restricted to the tangent bundle of M defines a pseudo-Riemannian metric on M. We denote by $TM^\perp$ the vector bundle over M whose fibers $T_xM^\perp$ are the orthogonal complements of T_xM in $T_x\overline{M}$. In particular, we have a direct sum of vector bundles $T\overline{M} = TM \oplus TM^\perp$.

Let us denote by ∇ and $\overline{\nabla}$ the Levi-Civita connections for M and $\overline{M}$, respectively. Suppose that X and Y are vector fields tangent to M defined in an open subset of some point $x \in M$. We can extend X and Y to vector fields $\overline{X}$ and $\overline{Y}$, respectively, defined in an open neighborhood of x in $\overline{M}$. If we compute $(\overline{\nabla}_{\overline{X}}\overline{Y})_x$, then by Lemma 1 in page 99 of [14] it follows that the value obtained depends only on X and Y. Hence, we will write $\overline{\nabla}_XY$ to denote the vector field thus obtained that is tangent to $\overline{M}$ but only defined in some open subset of M. Since we have a direct sum $T\overline{M} = TM \oplus TM^\perp$, we can decompose $\overline{\nabla}_XY$ into its component tangent to M and its component orthogonal to M. In other words we have

$$\overline{\nabla}_XY = (\overline{\nabla}_XY)^\top + (\overline{\nabla}_XY)^\perp,$$

where $(\cdot)^\top$ and $(\cdot)^\perp$ denote the orthogonal projections of $T\overline{M}$ onto TM and $TM^\perp$, respectively. It is well known (see Lemma 3 in page 99 of [14]) that the component tangent to M is the Levi-Civita of M. More precisely, we have

$$(\overline{\nabla}_XY)^\top = \nabla_XY$$

for every pair of vector fields X and Y tangent to M and defined in an open subset of M. On the other hand, the orthogonal component of $\overline{\nabla}_XY$ depends at every given point only on the values of X and Y at such point, thus defining a tensor field on the manifold M (see Lemma 4 in page 100 of [14]). Such tensor is called the second fundamental form of M in $\overline{M}$ and we will denote it with α. Hence we have

$$\alpha(X,Y) = (\overline{\nabla}_XY)^\perp,$$

for every pair of vector fields X and Y tangent to M and defined in an open subset of M. Note that the second fundamental form of M takes values in the bundle $TM^\perp$.

From the previous remarks, we have the following basic equation that relates the Levi-Civita connections of $\overline{M}$ and of M and the second fundamental form of the latter in the former

$$\overline{\nabla}_X Y = \nabla_X Y + \alpha(X, Y), \tag{4.1}$$

where X, Y are vector fields tangent to M defined in an open subset of M.

We are interested in those pseudo-Riemannian manifolds whose second fundamental form is invariant under covariant derivation, and thus we describe now how to compute it.

Let X and ξ be a tangent and a normal vector fields to M, respectively, both defined in an open subset of M. In other words, X and ξ are smooth sections of TM and $TM^\perp$, respectively, defined in an open subset of M. Following the same sort of arguments as used above, we can compute $\overline{\nabla}_X \xi$ as a vector field tangent to $\overline{M}$ defined on an open subset of M and depending only on X and ξ. Furthermore, we can also decompose $\overline{\nabla}_X \xi$ into a component tangent to M and one orthogonal to M. We will denote by $S_\xi(X)$ the component of $\overline{\nabla}_X \xi$ tangent to M and with $\nabla^\perp_X \xi$ the component of $\overline{\nabla}_X \xi$ orthogonal to M. This yields the expression

$$\overline{\nabla}_X \xi = S_\xi(X) + \nabla^\perp_X \xi, \tag{4.2}$$

It is known that the value of $S_\xi(X)$ at some x depends only on X_x (see Remark 39(3) in page 119 of [14]). Hence, for every normal vector field ξ to M and x in the domain of ξ we have a linear map $S_\xi : T_x M \to T_x M$. This defines a tensor S_ξ that is called the shape operator of M in $\overline{M}$ with respect to ξ. On the other hand, the assignment

$$(X, \xi) \mapsto \nabla^\perp_X \xi,$$

is a connection on the vector bundle $TM^\perp$, that is called the normal connection of M (see Definition 31 in page 114 of [14]).

The covariant derivative of the second fundamental form α is defined as follows

$$(\nabla_X \alpha)(Y, Z) = \nabla^\perp_X(\alpha(Y, Z)) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z),$$

where X, Y, Z are vector fields tangent to M. The value of $(\nabla_X \alpha)(Y, Z)$ at x depends only on X_x, Y_x, Z_x, and so it defines a tensor $\nabla\alpha$.

Definition 4.1. A pseudo-Riemannian submanifold M of a pseudo-Riemannian manifold $\overline{M}$ is called a parallel submanifold if its second fundamental form α is parallel, i.e. if $\nabla\alpha = 0$.

We recall that any parallel tensor (i.e. a tensor T such that $\nabla T = 0$) is invariant under parallel translation. Hence, a parallel tensor can be recovered from its value at a single point by parallel transport of such value. In particular, if M is a connected parallel submanifold, the second fundamental form of M is completely determined by its value at a single point.

Proposition 4.2. *Let $C : I \to \overline{M}$ be a curve parametrized by arc-length in a 2-dimensional Riemannian manifold $\overline{M}$. If $\kappa : I \to \mathbb{R}$ denotes the geodesic curvature of C, then we have*

$$\alpha(C'(t), C'(t)) = \kappa(t)N(t),$$

for every $t \in I$, where α is the second fundamental form of C in $\overline{M}$ and N is the unit normal to C in the direction of C''. In particular, C defines a parallel submanifold of $\overline{M}$ if and only if κ is constant.

Proof. The formula that relates α with κ is an easy consequence of the definitions of both of them. The last claim also follows easily by using the elementary fact that N is parallel with respect to $\nabla^{\perp}$. $\square$

Our main interest lies in the study of parallel Lagrangian submanifolds of complex hyperbolic spaces. These submanifolds have been completely classified in [9] and [10]. An important tool to study parallel Lagrangian submanifolds of complex hyperbolic spaces turns out to be their pull back under the Hopf fibration.

Definition 4.3. Let M be a Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$. The complete inverse of M is defined as $\widehat{M} = \pi^{-1}(M)$, where $\pi : \mathrm{H}^{2n+1}_1\mathbb{R} \to \mathrm{H}^n\mathbb{C}$ is the Hopf fibration associated to $\mathrm{H}^n\mathbb{C}$.

By Proposition 4.1 and Lemma 1.1 of [9] and the fact that $\mathrm{H}^{2n+1}_1\mathbb{R}$ is totally umbilical in $\mathbb{C}^{n+1}_1$, it follows that a Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$ and its complete inverse satisfy the following properties.

Theorem 4.4 ([9]). *Let M be a Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$ and $\widehat{M}$ its complete inverse in $\mathrm{H}^{2n+1}_1\mathbb{R}$. Then,*

1. *$\widehat{M}$ is Lagrangian as a submanifold of $\mathbb{C}^{n+1}_1$.*
2. *M is complete if and only if $\widehat{M}$ is complete.*
3. *M is parallel in $\mathrm{H}^n\mathbb{C}$ if and only if $\widehat{M}$ is parallel in $\mathrm{H}^{2n+1}_1\mathbb{R}$.*
4. *$\widehat{M}$ is parallel as a submanifold of $\mathrm{H}^{2n+1}_1\mathbb{R}$ if and only if it is parallel as a submanifold of $\mathbb{C}^{n+1}_1$.*

If $\widehat{M}$ is the complete inverse of a Lagrangian submanifold M of $\mathrm{H}^n\mathbb{C}$, then we denote by $\widehat{\alpha}$ the second fundamental form of $\widehat{M}$ in $\mathbb{C}^{n+1}_1$. In particular, the above implies that α is parallel (i.e. M is parallel in $\mathrm{H}^n\mathbb{C}$) if and only if $\widehat{\alpha}$ is parallel (i.e. $\widehat{M}$ is parallel in $\mathbb{C}^{n+1}_1$).

We say that two submanifolds M_1, M_2 of $\mathrm{H}^n\mathbb{C}$ are holomorphically (isometrically) congruent if there is a biholomorphism (respectively, an isometry) of $\mathrm{H}^n\mathbb{C}$ that maps M_1 onto M_2. Since any biholomorphism of $\mathrm{H}^n\mathbb{C}$ is an isometry, we have that any pair of holomorphically congruent submanifolds of $\mathrm{H}^n\mathbb{C}$ are also congruent as Riemannian submanifolds. In particular, two holomorphically congruent submanifolds have the same second fundamental form at corresponding points. More precisely, we have the following result.

Proposition 4.5. *If $f \in \mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ is a holomorphic congruence between two submanifolds M_1 and M_2, then for every $x \in M_1$ the map $df_x : T_x\mathrm{H}^n\mathbb{C} \to T_{f(x)}\mathrm{H}^n\mathbb{C}$ induces linear isometries $T_xM_1 \to T_{f(x)}M_2$ and $T_xM_1^\perp \to T_{f(x)}M_2^\perp$ such that*

$$\alpha_2(df_x(u), df_x(v)) = df_x(\alpha_1(u,v)),$$

for every $u, v \in T_xM_1$, where α_1 and α_2 denote the second fundamental forms of M_1 at x and M_2 at $f(x)$, respectively.

A remarkable fact proved in [9] and [10] is that for parallel Lagrangian submanifolds of $\mathrm{H}^n\mathbb{C}$ the converse of the above result is also true. To state such result, we consider first a certain replacement for the second fundamental form for Lagrangian submanifolds.

If M is a Lagrangian submanifold of a Kaehler manifold $\overline{M}$, then for every $x \in M$ the map:

$$\begin{aligned} T_xM &\to T_xM^\perp \\ u &\mapsto iu, \end{aligned}$$

is a linear isometry. In particular, if α is the second fundamental form of M in $\overline{M}$ and J denotes the complex structure on $\overline{M}$, then $\sigma = J\alpha$ is a tensor of type $(1,2)$ (i.e. 2-covariant and 1-contravariant, see [14] for further details on this notation). From the fact that the complex structure is parallel for a Kaehler manifold we obtain the following result.

Proposition 4.6. *Let M be Lagrangian submanifold of a Kaehler manifold $\overline{M}$. For α the second fundamental form of M in $\overline{M}$, let $\sigma = J\alpha$. Then*

$$\nabla\sigma = J\nabla\alpha.$$

In particular, M is a parallel submanifold of $\overline{M}$ if and only if σ is a tensor parallel in M.

Proof. Note that for tangent vector fields X, Y, Z on M we have

$$\begin{aligned} (\nabla\sigma)(X,Y,Z) &= \nabla_X(J\alpha(Y,Z)) - J\alpha(\nabla_XY,Z) - J\alpha(Y,\nabla_XZ) \\ J(\nabla\alpha)(X,Y,Z) &= J\nabla^\perp_X(\alpha(Y,Z)) - J\alpha(\nabla_XY,Z) - J\alpha(Y,\nabla_XZ). \end{aligned}$$

Hence, it suffices to show that

$$J\nabla^\perp_X\xi = \nabla_X(J\xi),$$

for vector fields X and ξ tangent and normal to M, respectively. Since $\overline{M}$ is Kaehler, then with respect to its connection $\overline{\nabla}$ and for the above X and ξ we have

$$J\overline{\nabla}_X(\xi) = \overline{\nabla}_X(J\xi),$$

and from equations (4.1) and (4.2) we obtain

$$J(S_\xi(X) + \nabla^\perp_X\xi) = \nabla_X(J\xi) + \alpha(X, J\xi).$$

Finally, using the fact that J reverses the tangent and normal directions to M and comparing such directions, we get $J\nabla^\perp_X\xi = \nabla_X(J\xi)$, as desired. $\square$

We now state the converse of Proposition 4.5 in terms of the tensor σ considered in Proposition 4.6. This result is essentially contained in [9] and [10]. The proof is basically an explanation of how to obtain it from the results of [9] and [10], which we include for the sake of completeness.

Theorem 4.7. *Let M_1 and M_2 be connected complete parallel Lagrangian submanifolds of $\mathrm{H}^n\mathbb{C}$ with complete inverses $\widehat{M_1}$ and $\widehat{M_2}$. For $j = 1, 2$, choose points $z_j \in M_j$, $\widehat{z}_j \in \widehat{M}_j$ and denote by σ_j, $\widehat{\sigma}_j$ the value of the tensor obtained from the second fundamental form for M_j, $\widehat{M}_j$ as submanifold of $\mathrm{H}^n\mathbb{C}$, $\mathbb{C}_1^{n+1}$, respectively, as considered in Proposition 4.6. Then the following conditions are equivalent.*

1. *M_1 and M_2 are holomorphically congruent in $\mathrm{H}^n\mathbb{C}$.*
2. *There exists a linear isometry $L : T_{z_1}M_1 \to T_{z_2}M_2$ for which we have $L \circ \sigma_1(\cdot,\cdot) = \sigma_2(L(\cdot), L(\cdot))$.*
3. *$\widehat{M_1}$ and $\widehat{M_2}$ are holomorphically congruent in $\mathbb{C}_1^{n+1}$ with respect to a complex linear map on $\mathbb{C}_1^{n+1}$.*
4. *There exists a linear isometry $L : T_{z_1}\widehat{M_1} \to T_{z_2}\widehat{M_2}$ for which we have $L \circ \widehat{\sigma}_1(\cdot,\cdot) = \widehat{\sigma}_2(L(\cdot), L(\cdot))$*

Proof. Lemma 6.1 of [10] proves that the bilinear form $\widehat{\sigma}_j$ and the inner product from $\mathbb{C}_1^{n+1}$ define a structure of an orthogonal Jordan algebra (OJA) on $T_{z_j}\widehat{M}_j$. We denote by $\mathfrak{A}_j$ such an OJA.

The equivalence of (1) and (4) above follows from parts (1) and (3) of Theorem 6.3 from [10] using the fact that our condition (4) is equivalent to the statement that the OJA's $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are isomorphic.

The equivalence of (3) and (4) above is explicitly stated in Remark 6.5 of [10]. Here again, (4) is stated in [10] in the language of OJA's.

For the equivalence of (2) and (4) we observe some geometric relations between a Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$ and its complete inverse. Let M be such a Lagrangian submanifold with complete inverse $\widehat{M}$; and let us choose $\widehat{z} \in \widehat{M}$. Further, let $\widehat{\sigma}$ and σ be the values at $\widehat{z}$ and z, respectively, of the tensor obtained as in Proposition 4.6 from the second fundamental form of $\widehat{M}$ and M, respectively. For such choices and as before, we consider $\widehat{M}$ as a submanifold of $\mathbb{C}_1^{n+1}$. Observe that the restriction of the Hopf fibration gives a map $\pi : \widehat{M} \to M$ that is still a principal fibration with structure group $\mathrm{U}_{n+1}(1)$. As observed on page 97 of [9], such restriction is a pseudo-Riemannian submersion. If we denote by $\mathcal{H}_{\widehat{z}}(\widehat{M})$ the horizontal subspace at $\widehat{z}$ with respect to this submersion, then we have the following properties obtained from equation (4.6) in [9]:

$$\text{If } u, v \in \mathcal{H}_{\widehat{z}}(\widehat{M}),\ \text{then } \widehat{\sigma}(u,v) \in \mathcal{H}_{\widehat{z}}(\widehat{M}) \text{ and } d\pi_{\widehat{z}}(\widehat{\sigma}(u,v)) = \sigma(d\pi_{\widehat{z}}(u), d\pi_{\widehat{z}}(u)).$$

$$\widehat{\sigma}(u, i\widehat{z}) = 2iu \text{ for every } u \in \mathcal{H}_{\widehat{z}}(\widehat{M}).$$

$$\widehat{\sigma}(i\widehat{z}, i\widehat{z}) = 0.$$

Since the vector $i\widehat{z}$ spans (over $\mathbb{R}$) the vertical subspace complementary to $\mathcal{H}_{\widehat{z}}(\widehat{M})$, it follows that σ determines the values of $\widehat{\sigma}$ in such a way that the equivalence of (2) and (4) is now clear. □

5. Curvature properties of the MASG orbits in $\mathrm{H}^n\mathbb{C}$.

We start this section with the following basic observation.

Proposition 5.1. *Let G be a connected Lie group acting smoothly on a manifold $\overline{M}$. Then, for every $x \in \overline{M}$, the orbit Gx is a smooth submanifold of $\overline{M}$. Moreover, if G is Abelian and preserves a pseudo-Riemannian metric in $\overline{M}$, and T_xGx is non-degenerate in $T_x\overline{M}$ (e.g. if $\overline{M}$ is Riemannian), then Gx is a pseudo-Riemannian submanifold and it is flat with the respect to the induced metric.*

Proof. If we denote by G_x the stabilizer of x in G, and we let

$$\begin{aligned} f : G &\to \overline{M} \\ x &\mapsto gx, \end{aligned}$$

be the orbit map at x, then there is an induced continuous and injective map $\tilde{f} : G/G_x \to \overline{M}$ such that $f = \tilde{f} \circ p$, where $p : G \to G/G_x$ is the quotient map. Since p defines a smooth fiber bundle (see Example 5.1 in page 55 of [5]), then there are smooth local sections of p in a neighborhood of every point of G/G_x. If s is any such section, then $\tilde{f} = f \circ s$ in the domain of s. This proves the smoothness of $\tilde{f}$.

Next, we observe that the kernel of df_e is $\mathfrak{g}_x$, the Lie algebra of G_x. Since $\mathfrak{g}_x$ is the kernel of dp_e, it follows that $d\tilde{f}_{eG_x}$ is injective. On the other hand, the map $\tilde{f}$ is clearly G-equivariant, and so it follows that $\tilde{f}$ has everywhere an injective differential. This implies that $\tilde{f}$ is an immersion and so its image Gx is a submanifold of $\overline{M}$.

We now assume that G is Abelian and preserves a metric on $\overline{M}$. Hence, G/G_x is a connected Abelian Lie group itself. Also, for T_xGx nondegenerate, the G-equivariance of $\tilde{f}$ and the G-invariance of the metric on $\overline{M}$ implies that the tangent space of Gx at every other point is nondegenerate as well. In particular, G/G_x is a pseudo-Riemannian submanifold of $\overline{M}$. Moreover, there is a pseudo-Riemannian metric on G/G_x so that the map $\tilde{f}$ is an isometry of G/G_x onto the orbit Gx. Again, the G-equivariance of $\tilde{f}$ implies that the metric on G/G_x is invariant under the group translations.

On the other hand, being a connected Abelian group, the universal covering space of G/G_x is $\mathbb{R}^k$, for some k. The invariant metric on G/G_x thus lifts to a metric on $\mathbb{R}^k$ that is translation invariant. But any such metric on $\mathbb{R}^k$ is clearly flat and so the metric on both G/G_x and the orbit Gx is flat. □

By Theorem 3.6 we obtain the following complete description of Abelian orbits in $\mathrm{H}^n\mathbb{C}$.

Theorem 5.2. *Let M be an Abelian orbit in* $\mathrm{H}^n\mathbb{C}$*, then there exists a biholomorphism* φ *from* $\mathrm{H}^n\mathbb{C}$ *onto either* $\mathbb{B}^n$ *or* D_n *such that* $\varphi(M)$ *is an orbit of a subgroup of one of the groups listed in Theorem* 3.6*.*

Proof. Let H be an Abelian subgroup of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ such that M is an H-orbit. Let H_1 be MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ that contains H. Hence, there is a biholomorphism φ onto either $\mathbb{B}^n$ or D_n with respect to which H_1 determines a pair analytically equivalent to a pair in the statement of Theorem 3.6. Denote by H_1' the subgroup for the latter pair. With respect to the homomorphism

$$\begin{aligned} H_1 &\to H_1' \\ h &\mapsto \varphi h \varphi^{-1} \end{aligned}$$

let H' be the image of H. Then, we observe that $\varphi(M)$ is an H'-orbit. □

We need the following well known results.

Lemma 5.3. *Let G be a connected Lie group acting smoothly on a manifold $\overline{M}$. If M is a G-orbit, then for every $x \in M$ we have*

$$T_xM = \{X_x^* : X \in \mathfrak{g}\},$$

where X^* is the vector field defined by

$$X_x^* = \frac{d}{dt}\bigg|_{t=0} (\exp(tX)x),$$

for every $X \in \mathfrak{g}$. In particular, if the G-orbit M has trivial stabilizers, then T_xM and $\mathfrak{g}$ are isomorphic as vector spaces.

Proof. It is an immediate consequence of Proposition 5.1. □

Lemma 5.4. *For $k < n$, the natural inclusion maps*

$$\begin{aligned} \mathbb{B}^k &\to \mathbb{B}^n, \\ w &\mapsto (0, w), \end{aligned} \qquad\qquad \begin{aligned} D_k &\to D_n \\ w &\mapsto (0, w) \end{aligned}$$

define totally geodesic holomorphic embeddings.

Proof. We will explain the proof for the embedding $\mathbb{B}^k \hookrightarrow \mathbb{B}^n$. The proof for the embedding of Siegel domains is similar.

As bounded symmetric domains and according to Table V in page 518 of [4], we have the realizations

$$\begin{aligned} \mathbb{B}^k &= \mathrm{SU}(k,1)/\mathrm{U}(k), \\ \mathbb{B}^n &= \mathrm{SU}(n,1)/\mathrm{U}(n). \end{aligned}$$

With respect to these, the embedding $\mathbb{B}^k \hookrightarrow \mathbb{B}^n$ from the statement corresponds to an embedding of Lie groups that induces the following embedding of their Lie algebras

$$\mathfrak{su}(k,1) \hookrightarrow \mathfrak{su}(n,1)$$
$$A \mapsto \begin{pmatrix} 0 & 0 \\ 0 & A \end{pmatrix}.$$

The structure of bounded symmetric domains in both $\mathbb{B}^k$ and $\mathbb{B}^n$ is determined by Cartan decompositions of $\mathfrak{su}(k,1)$ and $\mathfrak{su}(n,1)$, respectively, with fixed compact subalgebras given by $\mathfrak{u}(k)$ and $\mathfrak{u}(n)$, respectively (see Chapters VI and VIII of [4]). It is easy to see that the above embedding $\mathfrak{su}(k,1) \hookrightarrow \mathfrak{su}(n,1)$ preserves the corresponding Cartan decompositions, and so it follows that the (-1)-eigenspace of the Cartan decomposition of $\mathfrak{su}(k,1)$ embeds into the (-1)-eigenspace of the Cartan decomposition of $\mathfrak{su}(n,1)$. Hence, the result is a consequence of Theorem 7.2 of [4]. □

We now prove that all MASG orbits in $\mathrm{H}^n\mathbb{C}$ are Lagrangian in an appropiate submanifold.

Theorem 5.5. *Let M be a MASG orbit in $\mathrm{H}^n\mathbb{C}$ with real dimension k. Then there exists a totally geodesic Kaehler submanifold N of $\mathrm{H}^n\mathbb{C}$ that contains M as a Lagrangian submanifold. Furthermore, N is biholomorphic to $\mathrm{H}^k\mathbb{C}$.*

Proof. By taking the real part of the Hermitian metric in D_n as shown in Section 2, we obtain the following corresponding Riemannian metric for D_n

$$\begin{aligned} h_{D_n} = \frac{1}{\mathrm{Im}(z_n) - |z'|^2} \Bigg[& \frac{dx_n^2 + dy_n^2}{4(\mathrm{Im}(z_n) - |z'|^2)} + \sum_{k=1}^{n-1} (dx_k^2 + dy_k^2) \\ & + \frac{1}{\mathrm{Im}(z_n) - |z'|^2} \sum_{k=1}^{n-1} \mathrm{Re}(z_k)(dy_k \odot dx_n - dx_k \odot dy_n) \\ & - \frac{1}{\mathrm{Im}(z_n) - |z'|^2} \sum_{k=1}^{n-1} \mathrm{Im}(z_k)(dx_k \odot dx_n + dy_k \odot dy_n) \\ & + \frac{1}{\mathrm{Im}(z_n) - |z'|^2} \sum_{k,l=1}^{n-1} \mathrm{Re}(\overline{z}_k z_l)(dx_k \otimes dx_l + dy_k \otimes dy_l) \\ & - \frac{1}{\mathrm{Im}(z_n) - |z'|^2} \sum_{k,l=1}^{n-1} \mathrm{Im}(\overline{z}_k z_l)(dy_k \otimes dx_l - dx_k \otimes dy_l) \Bigg], \end{aligned} \tag{5.1}$$

where $\alpha \odot \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. We will use this expression below.

Let M be a MASG orbit in $\mathrm{H}^n\mathbb{C}$. By Theorem 5.2 there exists a biholomorphism φ from $\mathrm{H}^n\mathbb{C}$ onto either $\mathbb{B}^n$ or D_n such that $\varphi(M)$ is a MASG orbit of one of the subgroups enumerated in Theorem 3.6. Since such a biholomorphism preserves the Kaehlerian structures involved we can replace $\mathrm{H}^n\mathbb{C}$ with either $\mathbb{B}^n$ or D_n and M with an orbit of one of the subgroups listed in Theorem 3.6. We now proceed to consider each case separately.

M is a $\mathfrak{E}(n)$-orbit. We choose $z \in M$, then there exists $k \in \{0, 1, \dots, n\}$ and indices $j_1, \dots, j_k$ such that $z_j \neq 0$ precisely when $j \in \{j_1, \dots, j_k\}$. Note that the number k and the indices $j_1, \dots, j_k$ do not depend on our choice of z. This is easily seen from the expression of the $\mathfrak{E}(n)$-action. Moreover, since any permutation of coordinates defines a biholomorphism in $\mathbb{B}^n$, we can assume that $\{j_1, \dots, j_k\} = \{n-k+1, \dots, n\}$, i.e. the last k coordinate indices of $\mathbb{B}^n$. Then M is a submanifold of $\mathbb{B}^k$ embedded in $\mathbb{B}^n$ by the map from Lemma 5.4. Hence, it is enough to show that M is a Lagrangian submanifold of $\mathbb{B}^k$, and so we will restrict the following discussion to the last k coordinates given by $\mathbb{B}^k$. In particular, M can be considered as a $\mathfrak{E}(k)$-orbit for the $\mathfrak{E}(k)$-action on $\mathbb{B}^k$.

Since for $z \in M \subset \mathbb{B}^k$ we have $z_j \neq 0$ for every j, then the $\mathfrak{E}(k)$-action has trivial stabilizers at every point in M. By Lemma 5.3, the orbit M is a (real) k-dimensional submanifold of $\mathbb{B}^k$. Hence, we need to show that $T_z M$ and $iT_z M$ are orthogonal for every $z \in M$. But since M, the complex structure, and the Hermitian metric are all invariant under the $\mathfrak{E}(k)$-action, it is enough to check such orthogonality at a single point. We now observe that M has some point x lying in $\mathbb{R}^k_+$ and a direct computation using Lemma 5.3 shows that

$$T_x M = i\mathbb{R}^k, \quad iT_x M = \mathbb{R}^k,$$

and these are easily seen to be orthogonal for the Riemannian metric of $\mathbb{B}^k$. We recall that this Riemannian metric is a multiple of the real part of the Hermitian metric whose expression is given in Section 2.

M is a $\mathfrak{P}(n)$-orbit. Choose $z \in M$ with coordinates $z = (z', z_n)$ corresponding to those in the Seigel domain D_n. We now look at z' and use the above arguments to single out the nonvanishing coordinates of z'. By doing so, we can consider that M is a submanifold of D_k embedded in D_n by the map in Lemma 5.4, in such a way that M is a $\mathfrak{P}(k)$-orbit in D_k. Using the expression of the $\mathfrak{P}(k)$-action on D_k we also conclude that the stabilizers of such action on M are trivial, thus implying that M is a k-dimensional submanifold of D_k.

Hence, we need to prove that $T_z M$ and $iT_z M$ are orthogonal for every $z \in M$. But again, since the involved structures are $\mathfrak{P}(k)$-invariant we only have to check this at a single point. Now observe that in M there is some point of the form (x', iy_k) where $x' \in \mathbb{R}^{k-1}_+$ and $y_k \in \mathbb{R}$. For such a choice, applying Lemma 5.3 to the $\mathfrak{P}(k)$-action we obtain:

$$T_{(x', iy_k)} M = i\mathbb{R}^{k-1} \oplus \mathbb{R}$$
$$iT_{(x', iy_k)} M = \mathbb{R}^{k-1} \oplus i\mathbb{R}.$$

Let us now choose $u \in i\mathbb{R}^{k-1} \oplus \mathbb{R}$ and $v \in \mathbb{R}^{k-1} \oplus i\mathbb{R}$, that are then of the form:

$$u = (iu_1, \dots, iu_{k-1}, u_k)$$
$$v = (v_1, \dots, v_{k-1}, iv_k),$$

where $u_j, v_j \in \mathbb{R}$ for every j.

The following analysis of the terms inside the brackets of equation (5.1) prove that $h_{D_k}(u,v) = 0$.

- The first and second terms inside the brackets of equation (5.1) vanish when evaluated at (u,v) by the orthogonality of $\mathbb{R}$ and $i\mathbb{R}$ in $\mathbb{C}$ with the usual flat Riemannian metric.
- We observe that by the choice of u, v we have $dx_k(v) = dy_k(u) = dx_j(u) = dy_j(v) = 0$ for every $j = 1, \dots, k-1$. From this it follows easily that $dy_j \odot dx_k(u,v) = dx_j \odot dy_k(u,v) = 0$, for every $j = 1, \dots, k-1$. This implies that the third term in equation (5.1) vanishes at (u,v).
- The fourth and sixth terms in equation (5.1) vanish because $\mathrm{Im}(z_r) = \mathrm{Im}(\overline{z}_r z_s) = 0$ for every $r, s = 1, \dots, k-1$, since $z_j = x_j \in \mathbb{R}$ if $1 \le j \le k-1$.
- Finally the fifth term in equation (5.1) vanishes at (u,v) since one can check directly that $dx_r \otimes dx_s(u,v) = dy_r \otimes dy_s(u,v) = 0$, for every $r, s = 1, \dots, k-1$.

This analysis shows that M is Lagrangian in D_k.

M is a $\mathfrak{H}(n)$-orbit. The same sort of argument used above for the parabolic case allows us to assume that M is a submanifold of D_k embedded in D_n by the map in Lemma 5.4, in such a way that it is a (real) k-dimensional $\mathfrak{H}(k)$-orbit in D_k for which the stabilizers are trivial. We now have to show that M is Lagrangian in such D_k and, as before, we only need to check the orthogonality condition at a single point.

We now claim that M has a point of the form (x', z_k) with $x' \in \mathbb{R}_+^{k-1}$ and $\mathrm{Im}(z_k) - |x'|^2 = 1$. First we note that by our assumptions, $z \in M$ implies that $z_j \neq 0$ for every $j = 1, \dots, k-1$. If we choose $z = (z', z_k) \in M$, then there is some $t \in \mathbb{T}^{k-1}$ such that $(t,1)(z', z_k) = (x', z_k)$, where $x' \in \mathbb{R}_+^{k-1}$. On the other hand, $r = \mathrm{Im}(z_k) - |x'|^2 > 0$, and the element $(1, r^{-1/2})(x', z_k) = (r^{-1/2}x', r^{-1}z_k)$ is easily seen to satisfy the required properties.

At (x', z_k) given as above we have

$$\begin{aligned} T_{(x',z_k)}M &= i\mathbb{R}^{k-1} \oplus \mathbb{R}(x', 2z_k) \\ iT_{(x',z_k)}M &= \mathbb{R}^{k-1} \oplus \mathbb{R}(ix', 2iz_k). \end{aligned}$$

Let us consider vectors

$$\begin{aligned} u &= (a_k x' + ia', 2a_k z_k) \in T_{(x',z_k)}M \\ v &= (b' + ib_k x', 2ib_k z_k) \in iT_{(x',z_k)}M \end{aligned}$$

where $(a', a_k), (b', b_k) \in \mathbb{R}^{k-1} \times \mathbb{R}$. By using equation (5.1) and the fact that $\mathrm{Im}(z_k) - |x'|^2 = 1$ we obtain

$$\begin{aligned}
h_{D_k}(u, v) = \frac{1}{4}&(-4a_k x_k b_k y_k + 4a_k y_k b_k x_k) + a_k(b' \cdot x') + b_k(a' \cdot x') \\
&+ \sum_{j=1}^{k-1} x_j(-a_j b_k y_k + a_k x_k b_k x_j - (a_k x_j b_k x_k + a_k b_j y_k)) \\
&+ \sum_{r,s=1}^{k-1} x_r x_s(a_k b_s x_r + a_r b_k x_s) \\
&= a_k(b' \cdot x') + b_k(a' \cdot x') - y_k \sum_{j=1}^{k-1} x_j(a_j b_k + a_k b_j) \\
&+ \sum_{r,s=1}^{k-1} x_r x_s(a_k b_s x_r + a_r b_k x_s) \\
&= a_k(b' \cdot x') + b_k(a' \cdot x') - y_k b_k(a' \cdot x') - y_k a_k(b' \cdot x') \\
&+ a_k(b' \cdot x')|x'|^2 + b_k(a' \cdot x')|x'|^2 \\
&= b_k(a' \cdot x')(1 - y_k + |x'|^2) + a_k(b' \cdot x')(1 - y_k + |x'|^2),
\end{aligned}$$

which vanishes since $y_k - |x'|^2 = 1$. This implies that M is Lagrangian in D_k.

M is a $\mathfrak{N}(n)$-orbit. In this case the action $\mathfrak{N}(n)$ has trivial stabilizers on all of D_n. This implies that M is a (real) n-dimensional submanifold of D_n. As before we only need to check the required orthogonality condition at some point. It is easy to check that M has a point of the form $(iy', iy_n) \in i\mathbb{R}^n$ and at such a point we obtain using Lemma 5.3

$$\begin{aligned}
T_{(iy', iy_n)}\mathcal{O} &= \mathbb{R}^n \\
iT_{(iy', iy_n)}\mathcal{O} &= i\mathbb{R}^n.
\end{aligned}$$

Let us now choose $u \in \mathbb{R}^n$ and $v \in i\mathbb{R}^n$, that can be written as

$$\begin{aligned}
u &= (u_1, \dots, u_n) \\
v &= (iv_1, \dots, iv_n),
\end{aligned}$$

where $u_k, v_k \in \mathbb{R}$ for every k.

The following analysis of the terms inside the brackets of equation (5.1) prove that $h_{D_n}(u, v) = 0$.

- The first and second terms inside the brackets of equation (5.1) vanish when evaluated at (u, v) by the orthogonality of $\mathbb{R}$ and $i\mathbb{R}$ in $\mathbb{C}$ with the usual flat Riemannian metric.
- The third and sixth terms in equation (5.1) vanish because $\mathrm{Re}(z_k) = \mathrm{Im}(\overline{z}_k z_l) = 0$ for every k, l, since $z_k = iy_k \in i\mathbb{R}$ for every k.

- We observe that by the choice of u, v we have $dx_k(v) = dy_k(u) = 0$ for every k. From this it follows that $dx_k \odot dx_n(u,v) = dy_k \odot dy_n(u,v) = 0$, for every k. This implies that the fourth term in equation (5.1) vanishes at (u,v).
- Finally the fifth term in equation (5.1) vanishes at (u,v) since one can check directly that $dx_k \otimes dx_l(u,v) = dy_k \otimes dy_l(u,v) = 0$, for every k, l.

Again, we conclude that M is Lagrangian in D_n.

M is a $\mathfrak{N}(n,k)$-orbit. For this case, we decompose the points $z \in D_n$ with the expression $z = (z', w', z_n) \in \mathbb{C}^k \times \mathbb{C}^{n-k-1} \times \mathbb{C}$. As in the cases before the nilpotent group action, we have to look at the first k coordinates that are acted upon by the torus $\mathbb{T}^k$. If j of such coordinates are non zero for some (and hence any) point in M, then we may assume that they are precisely the last j coordinates among those of z' in the expression (z', w', z_n). In this case, by applying the same reductions as before we can assume that M is a submanifold of D_{n-k+j} and that it is also a $\mathfrak{N}(n-k+j, j)$-orbit with trivial stabilizers. In particular, M is a (real) $(n-k+j)$-submanifold of D_{n-k+j}. For simplicity we will denote $m = n-k+j$.

Again, we will check the required orthogonality property at a single point. In this case, we observe that M has a point of the form $(x', iy', iy_m) \in \mathbb{R}^j \times i\mathbb{R}^{m-j-1} \times i\mathbb{R}$, and at such a point

$$\begin{aligned} T_{(x',iy',iy_m)}M &= i\mathbb{R}^j \oplus \mathbb{R}^{m-j} \\ iT_{(x',iy',iy_m)}M &= \mathbb{R}^j \oplus i\mathbb{R}^{m-j}. \end{aligned}$$

Let us now choose $u \in i\mathbb{R}^j \oplus \mathbb{R}^{m-j}$ and $v \in \mathbb{R}^j \oplus i\mathbb{R}^{m-j}$, that can be written as

$$\begin{aligned} u &= (iu_1, \dots, iu_j, u_{j+1}, \dots, u_m) \\ v &= (v_1, \dots, v_j, iv_{j+1}, \dots, iv_m), \end{aligned}$$

where $u_r, v_r \in \mathbb{R}$ for every r.

The following analysis of the terms inside the brackets of equation (5.1) prove that $h_{D_n}(u,v) = 0$. To apply equation (5.1) we will refer to r and s as the summation indices.

- The first and second terms inside the brackets of equation (5.1) vanish when evaluated at (u,v) by the orthogonality of $\mathbb{R}$ and $i\mathbb{R}$ in $\mathbb{C}$ with the usual flat Riemannian metric.
- For the third term we have two cases to consider according to the value of the summation index r
  - The terms corresponding to $r \geq j+1$ vanish since in this case we have $\mathrm{Re}(z_r) = \mathrm{Re}(iy_r) = 0$.
  - For the terms corresponding to $r \leq j$, we observe that $dx_m(v) = dy_m(u) = dx_r(u) = dy_r(v) = 0$. From this it follows easily that $dy_r \odot dx_m(u,v) = dx_r \odot dy_m(u,v) = 0$, for $r \leq j$. Hence, the corresponding term vanish at (u,v).
- For the fourth term we have a similar situation according to the values of the summation index r.
  - If $r \leq j$, then $\mathrm{Im}(z_r) = \mathrm{Im}(x_r) = 0$, and the corresponding terms vanish.

  - For the terms corresponding to $r \geq j+1$, we now observe that $dx_m(v) = dy_m(u) = dx_r(v) = dy_r(u) = 0$. From this it follows easily that $dx_r \odot dx_m(u,v) = dy_r \odot dy_m(u,v) = 0$, for $r \geq j+1$. Hence, the corresponding term vanish at (u,v).
- For the fifth term, let r, s be the summation indices for equation (5.1). We now have two cases according to whether or not r, s both belong to the same of the two intervals $\{1, \dots, j\}$ and $\{j+1, \dots, m\}$.
  - If r, s do not belong to the same interval, then $\overline{z}_r z_s$ is pure imaginary and so the corresponding terms vanish.
  - If r, s belong to the same interval, then one can directly verify that by the choices of u, v we have $dx_r \otimes dx_s(u,v) = dy_r \otimes dy_s(u,v) = 0$.
- Finally, for the sixth term in equation (5.1) we also have two cases according to whether or not r, s belong to the same intervals as above.
  - If r, s belong to the same interval, then $\overline{z}_r z_s$ is real and the corresponding terms vanish.
  - If r, s do not belong to the same interval, then one can directly verify that $dy_r \otimes dx_s(u,v) = dx_r \otimes dy_s(u,v) = 0$, for our choices of u, v.

Then M is a Lagrangian submanifold of D_m. □

Using the previous result it is now easy to conclude that MASG orbits in the complex hyperbolic space are parallel.

Theorem 5.6. *Every MASG orbit in* $\mathrm{H}^n\mathbb{C}$ *is a complete parallel submanifold.*

Proof. Let M be an orbit in $\mathrm{H}^n\mathbb{C}$ of a MASG H of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. The completeness of M follows from homogeneity and the well known fact that a homogeneous Riemannian manifold is complete. By Theorem 5.5, there is a totally geodesic submanifold N of $\mathrm{H}^n\mathbb{C}$ biholomorphic to some $\mathrm{H}^k\mathbb{C}$ that contains M as a Lagrangian submanifold. Furthermore, by the proof of Theorem 5.5, M is still a MASG orbit in the corresponding $\mathrm{H}^k\mathbb{C}$. Then, since the property of being parallel does not change when we pass to a totally geodesic submanifold (see Lemma 1.1 of [9]), we can assume that $k = n$, i.e. that M is Lagrangian in $\mathrm{H}^n\mathbb{C}$. This implies that, if α is the second fundamental form of M, then we can consider $\sigma = i\alpha$, which is a tensor of type $(1,2)$ on M. By Proposition 4.6, it is enough to show that σ is parallel as a tensor in M.

Since H acts by isometries on $\mathrm{H}^n\mathbb{C}$ and preserves M, it also preserves its second fundamental form M. But H also preserves the complex structure and so it preserves the tensor σ on M.

On the other hand, by Proposition 5.1 and the proof of Theorem 5.5, the orbit map

$$\begin{aligned} H &\to M \\ h &\mapsto hz, \end{aligned}$$

is an H-equivariant diffeomorphism. From the proof of Proposition 5.1 it is also an isometry for some H-invariant metric on H. The pull-back of σ with respect to such orbit map thus defines an H-invariant tensor, which is then parallel in H. This last claim follows from the fact that in an Abelian Lie group with invariant metric a tensor is parallel if and only if it is (translation) invariant (see Exercise 6(ii) in Chapter II of [4]). Hence, the fact that the above orbit map is an isometry onto M implies that σ is parallel in M. $\square$

The properties obtained from Proposition 5.1 and Theorems 5.5 and 5.6 allow us to conclude the following result.

Theorem 5.7. *Let H be a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. Then for every H-orbit M with real dimension k there exists a totally geodesic Kaehler submanifold N of $\mathrm{H}^n\mathbb{C}$ biholomorphic to $\mathrm{H}^k\mathbb{C}$ containing M as a flat parallel Lagrangian submanifold. Moreover, if we denote by $\mathrm{H}^n\mathbb{C}_H$ the set of points in $\mathrm{H}^n\mathbb{C}$ where the H-action is free, then $\mathrm{H}^n\mathbb{C}_H$ is a connected open conull H-invariant subset of $\mathrm{H}^n\mathbb{C}$ where the real dimension of the H-orbits is n and, in particular, in this open subset the H-orbits are flat parallel Lagrangian submanifolds of $\mathrm{H}^n\mathbb{C}$.*

Proof. All but the last claim is a consequence of the previous results. To obtain the last claim, by Theorem 3.6 we can assume that H is one of the groups listed in its statement. From this, it is easy to see that $\mathrm{H}^n\mathbb{C}_H$ is an open conull subset of $\mathrm{H}^n\mathbb{C}$ by considering each case separately. In fact, for each case the action fails to be free in a union of a finite number of proper complex subspaces of $\mathbb{C}^{n+1}$ intersected with either $\mathbb{B}^n$ or D_n. $\square$

By Naitoh's classification of parallel totally real submanifolds of $\mathrm{H}^n\mathbb{C}$ ([9] and [10]) it follows that the converse is also true. In other words, every flat parallel totally real submanifold of $\mathrm{H}^n\mathbb{C}$ is a MASG orbit. This is essentially contained in [9] and [10] and their references. In the next theorem we state the result and in its proof we explain how to obtain it from these previous works. For this, we will use the following result.

Lemma 5.8. *Let M be a complete parallel Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$ and $\widehat{M}$ its complete inverse. Denote by $\pi : \widehat{M} \to M$ the pseudo-Riemannian submersion obtained by restricting the Hopf fibration of $\mathrm{H}^n\mathbb{C}$. Then, the fundamental tensors of $\pi : \widehat{M} \to M$ are both zero. In particular, M is flat if and only if $\widehat{M}$ is flat.*

Proof. Let us denote by $\widehat{\nabla}$ and ∇ the Levi-Civita connections of $\widehat{M}$ and M, respectively. Also, denote by V the vertical vector field whose flow is given by the $U_{n+1}(1)$-action. Then by equation (4.6) in page 98 of [9] we have for X and Y horizontal vector fields on $\widehat{M}$ the identities

$$\begin{aligned}
\widehat{\nabla}_X Y &= \mathsf{h}(\nabla_{d\pi(X)} d\pi(Y)) \\
\widehat{\nabla}_X V &= \mathcal{V}(\nabla^N_X V) \\
\widehat{\nabla}_V X &= \widehat{\nabla}_V V = 0,
\end{aligned}$$

where ∇^N denotes the Levi-Civita connection of $\mathrm{H}^{2n+1}_1\mathbb{R}$ and $\mathsf{h}(\nabla_{d\pi(X)} d\pi(Y))$ denotes the horizontal lift of $\nabla_{d\pi(X)} d\pi(Y)$. If T and A denote the fundamental tensors of the pseudo-Riemannian submersion $\pi : \widehat{M} \to M$, then by its definition as given in Section 2 it follows that $T = A = 0$. The last claim is now a consequence of Proposition 2.1 and the fact that the fibers are flat since they are one-dimensional. □

Theorem 5.9. *Let M be a complete flat parallel totally real submanifold of* $\mathrm{H}^n\mathbb{C}$*. Then there exists an Abelian subgroup H of* $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ *such that M is an H-orbit. In particular, every flat parallel Lagrangian submanifold of* $\mathrm{H}^n\mathbb{C}$ *is a MASG orbit.*

Proof. Given the first part of the statement, the last claim is an easy consequence of the fact that the MASG's of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ are n-dimensional.

Let M be a flat parallel totally real submanifold of $\mathrm{H}^n\mathbb{C}$. By Theorem 2.4 of [9] it follows that M is a Lagrangian submanifold of a Kaehler totally geodesic submanifold N of $\mathrm{H}^n\mathbb{C}$ biholomorphic to $\mathrm{H}^k\mathbb{C}$ where k is the real dimension of M. We recall from the theory of symmetric spaces (see Sections 5 and 7 in Chapter IV of [4]) that for a totally geodesic isometric embedding $\varphi : X \hookrightarrow \overline{X}$ of symmetric spaces, there is a homomorphism $\rho : \mathrm{Iso}_0(X) \to \mathrm{Iso}_0(\overline{X})$ of the connected components of their groups of isometries with respect to which φ is ρ-equivariant. In particular, the elements of every Abelian group of isometries of X extend to isometries of $\overline{X}$ to form an Abelian group. From this it follows that every Abelian group of biholomorphisms of N extends to an Abelian group of biholomorphisms of $\mathrm{H}^n\mathbb{C}$. This is a consequence that the group of biholomorphisms of a complex hyperbolic space is precisely the connected component of its group of isometries. In particular, if we prove that M is an Abelian orbit in N, then it will follow that it is an Abelian orbit in $\mathrm{H}^n\mathbb{C}$. From this discussion it follows that we can assume that M is a Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$.

As before, let us denote by $\widehat{M}$ the complete inverse of M. As observed in Lemma 5.8, the Hopf fibration above $\mathrm{H}^n\mathbb{C}$ restricts to a pseudo-Riemannian submersion $\widehat{M} \to M$ with a one-dimensional fiber. Theorem 2.2 also implies that the linear action of $\mathrm{U}(n,1)$ on $\mathbb{C}^{n+1}_1$ descends to the action of $\mathrm{PU}(n,1)$ on $\mathrm{H}^n\mathbb{C}$ that defines the group of biholomorphisms of $\mathrm{H}^n\mathbb{C}$. It follows that if $\widehat{M}$ is an orbit of an Abelian subgroup of $\mathrm{U}(n,1)$, then M is an orbit of an Abelian subgroup of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. In other words, it suffices to show that $\widehat{M}$ is an orbit of an Abelian group of unitary complex linear transformations of $\mathbb{C}^{n+1}_1$.

Naitoh's classification of complete parallel submanifolds of $\mathrm{H}^n\mathbb{C}$ ([9] and [10]) is based in associating to the corresponding complete inverses certain algebraic objects. More precisely, such classification associates to any parallel Lagrangian submanifold an orthogonal Jordan algebra (OJA as in the proof Theorem 4.7) and a Hermitian symmetric graded Lie algebra (HSGLA) satisfying suitable axioms. In fact, it is proved in [9] and [10] that there is a one-to-one correspondence between parallel Lagrangian submanifolds and a family of such algebraic objects satisfying a certain set of axioms. We will use some features of this correspondence and refer to [9] and [10] for further details. Nevertheless, while doing this, we will be careful to provide precise references to the results in Naitoh's work.

Following [10], we call a complete inverse of a parallel Lagrangian submanifold of $\mathrm{H}^n\mathbb{C}$ indecomposable if and only if its associated OJA in the above mentioned correspondence is indecomposable in the sense of Section 7 from [10]. Then by Remark 7.6 from [10] it follows that $\widehat{M}$ given above is, up to a unitary linear congruence, a product of indecomposable complete inverses. More precisely, there exist complete parallel Lagrangian submanifolds $\widehat{M_j} \subset \mathbb{C}^{k_j}_{\epsilon_j}$, where $j = 1\dots,l$, such that $n+1 = k_1 + \cdots + k_l$ and for which there is a unitary complex linear transformation

$$T : \mathbb{C}^{n+1}_1 \to \mathbb{C}^{k_1}_{\epsilon_1} \times \cdots \times \mathbb{C}^{k_l}_{\epsilon_l}$$

that maps $\widehat{M}$ onto $\widehat{M_1} \times \cdots \times \widehat{M_l}$. Here, and according to our previous notation, $\epsilon_j \in \{0,1\}$ and $\mathbb{C}^k_0$ is the k-dimensional complex vector space with the usual Hermitian form while $\mathbb{C}^k_1$ is the same vector space with the Hermitian form with signature $(k-1,1)$. Also, the target of the unitary map T carries the product Hermitian structure.

Furthermore, by Remark 7.11 of [10] we can assume that the following properties are satisfied

- $\epsilon_j = 1$ for just one j,
- $\widehat{M_j}$ is the complete inverse of a parallel Lagrangian submanifold M_j of either $\mathrm{H}^{k_j-1}\mathbb{C}$ or $\mathrm{P}^{k_j-1}\mathbb{C}$,
- $\widehat{M_j}$ is indecomposable.

We note here that the correspondence between OJA's and HSGLA's with parallel submanifolds of $\mathrm{H}^n\mathbb{C}$ mentioned above is in fact considered in [9] and [10] in such a way that it includes parallel submanifolds of $\mathrm{P}^n\mathbb{C}$. This is actually needed to complete the classification of parallel submanifolds in $\mathrm{H}^n\mathbb{C}$, and with this respect the results found in [11] play an important role.

By the previous discussion, it is enough to show that each submanifold $\widehat{M_j}$ is an orbit of an Abelian group of unitary transformations of $\mathbb{C}^{k_j}_{\epsilon_j}$. To achieve this we observe that by Lemma 5.8 the flatness of M implies that $\widehat{M}$ is flat as well. But then, since each $\widehat{M_j}$ is a factor of $\widehat{M}$, we also have that $\widehat{M_j}$ is flat. Hence, one more application of Lemma 5.8 allows us to assume that each M_j is flat.

In other words, each $\widehat{M_j}$ is indecomposable and the complete inverse of a flat parallel Lagrangian submanifold of either $\mathrm{H}^{k_j-1}\mathbb{C}$ or $\mathrm{P}^{k_j-1}\mathbb{C}$. The classification of parallel submanifolds from [9] and [10] is stated by providing a fairly complete description of all such complete inverses. This description also uses the constructions considered in [11]. Hence, the above conditions on the submanifolds $\widehat{M_j}$ together with Remark 7.11, Theorem 8.5(4) and Lemma 9.1 of [10] and Theorem 2.1(5) of [11] we find that each $\widehat{M_j}$ is given by one of the following possibilities.

Case 1: The complete inverse $\widehat{M} \subset \mathbb{C}^2_1$ of a closed curve in $\mathrm{H}^1\mathbb{C}$ with constant geodesic curvature.

Case 2: The complete inverse $\widehat{M} \subset \mathbb{C}^1_1$ of the one point trivial space $\mathrm{H}^0\mathbb{C}$.

Case 3: The complete inverse $\widehat{M} \subset \mathbb{C}_1^{k+1}$ of a parallel submanifold of $\mathrm{H}^k\mathbb{C}$, where the associated HSGLA is almost nilpotent as defined in page 135 of [10].

Case 4: The complete inverse $\widehat{M} \subset \mathbb{C}_0^2$ of a closed curve in $\mathrm{P}^1\mathbb{C}$ with constant geodesic curvature.

We now proceed to consider these possibilities and show that each one defines a complete inverse that is the orbit of an Abelian group of unitary complex linear transformations.

Case 1: It was proved in [3] that the closed curves in $\mathrm{H}^1\mathbb{C}$ with constant geodesic curvature are all given as orbits of one-parameter subgroups of $\mathrm{PU}(1,1)$. Such subgroups are conjugate to either the subgroups of $\mathrm{Aut}(D_1)$ given by

$$\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in \mathbb{R} \right\}, \quad \left\{ \begin{bmatrix} r & 0 \\ 0 & r^{-1} \end{bmatrix} : r \in \mathbb{R}_+ \right\},$$

or the subgroup of $\mathrm{Aut}(\mathbb{B}^1)$ given by

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t \in \mathbb{T} \right\}.$$

From this and Theorem 2.2 it follows that $\widehat{M}$ is an orbit of one of the following subgroups of $\mathrm{U}(1,1)$

$$\left\{ \begin{pmatrix} t & a \\ 0 & t \end{pmatrix} : a \in \mathbb{R}, t \in \mathbb{T} \right\},$$
$$\left\{ \begin{pmatrix} tr & 0 \\ 0 & tr^{-1} \end{pmatrix} : r \in \mathbb{R}_+, t \in \mathbb{T} \right\},$$
$$\left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} : t_1, t_2 \in \mathbb{T} \right\},$$

all of which are Abelian.

Case 2: This is rather a trivial case. It is enough to observe that $\mathrm{H}_1^1\mathbb{R}$ is simply the circle in $\mathbb{C}_1^1$ and the complete inverse $\widehat{M}$ coincides with such circle. Hence, $\widehat{M}$ is clearly an orbit of $\mathrm{U}(1)$ which is Abelian.

Case 3: This requires to understand the corresponding construction from [10]. The HSGLA that one associates to this case has a decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, where $\mathfrak{k}$ is a Lie subalgebra and $\mathfrak{p}$ is an $\mathrm{ad}(\mathfrak{k})$-invariant subspace. As explained in [9] and [10], the subspace $\mathfrak{p}$ has a Hermitian structure preserved by the group $K = \mathrm{Ad}_{\mathfrak{p}}(\mathfrak{k})$ and such that the manifold $\widehat{M}$ is equivalent to a K-orbit $K(\nu)$ in $\mathfrak{p}$ for some $\nu \in \mathfrak{p}$. More precisely, there is a unitary complex linear map $\mathbb{C}_1^{k+1} \to \mathfrak{p}$ that maps $\widehat{M}$ onto $K(\nu)$ (see Section 5 from [9] and Section 6 from [10]). Hence, it suffices to show that K is Abelian in this case. We will do so by proving that $\mathfrak{k}$ is an Abelian Lie algebra.

By the remarks in page 103 from [9], it follows that there is a real vector space V and a bilinear map $L : V \times V \to \mathrm{End}(V)$ such that $\mathfrak{g}$ is a subset of $V \oplus \widetilde{L} \oplus V$, where $\widetilde{L}$ is the real vector space generated by the maps of the form $L(u, v)$ with

$u, v \in V$. Furthermore, the Lie brackets are given by the formula

$$[(u_1, F, u_2), (v_1, G, v_2)] = \\ (F(v_1) - G(u_1), [F, G] - \frac{1}{2}L(u_1, v_2) + \frac{1}{2}L(v_1, u_2), G^t(u_2) - F^t(v_2)), \quad (5.2)$$

where A^t denotes the transpose of $A \in \mathrm{End}(V)$ with respect to a suitably defined inner product in V.

By Lemma 5.1 from [9], the Lie subalgebra $\mathfrak{k}$ is generated by the subset of triples of the form $(u, L(v_1, v_2) - L(v_2, v_1), u)$ for $u, v_1, v_2 \in V$. From the correspondences considered in [10], V admits a Jordan algebra structure with product $(u, w) \mapsto u \cdot w$ such that

$$L(u, v) = T_{u \cdot v} + [T_u, T_v]$$

for every $u, v \in V$, where T_u denotes the map $w \mapsto u \cdot w$ (see equation (5.12) in page 112 of [9] and the proof of Lemma 6.1 from [10]). In fact, with such structure, V turns out to be the OJA associated to $\widehat{M}$. From the above relations it follows that

$$L(u, v) - L(v, u) = T_{u \cdot v} + [T_u, T_v] - T_{v \cdot u} - [T_v, T_u] = 2[T_u, T_v].$$

At this point Lemma 8.2(2) from [10] implies that the latter vanishes. In other words, $L(u, v) - L(v, u) = 0$ for every $u, v \in V$. Hence, by the above remarks we conclude that

$$\mathfrak{k} = \{(u, 0, u) : u \in V\},$$

and so formula (5.2) implies that $\mathfrak{k}$ is Abelian.

Case 4: By following arguments similar to those from [3] one can show that the closed curves in $\mathrm{P}^1\mathbb{C}$ with constant geodesic curvature are precisely the orbits of subgroups conjugate to the group

$$\left\{ \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix} : t \in \mathbb{T} \right\}.$$

Then, using the Hopf fibration for $\mathrm{P}^1\mathbb{C}$ we conclude that $\widehat{M}$ is the orbit of a subgroup of $\mathrm{U}(2)$ conjugate to the one given by

$$\left\{ \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix} : t_1, t_2 \in \mathbb{T} \right\},$$

which is Abelian. □

6. Foliations defined by MASG's of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$.

Our main goal in this section is to prove that every MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ defines a pair of foliations with distinguished geometry. We show as well that the inverse statement is also true, i.e., that each pair of foliations possessing this distinguished geometry is originated from a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. To do so, we briefly discuss first some notions of foliations and their geometry. We refer to [15] and [7] for further details. We also obtain some results on isometric actions of Lie groups.

A foliation on a manifold $\overline{M}$ is a partition of $\overline{M}$ into connected submanifolds of the same dimension that can locally be given by the fibers of a submersion. The precise definition is as follows.

On a smooth manifold $\overline{M}$ a codimension q foliated chart is a pair (φ, U) given by an open subset U of $\overline{M}$ and a smooth submersion $\varphi : U \to V$, where V is an open subset of $\mathbb{R}^q$. For a foliated chart (φ, U) the connected components of the fibers of φ are called the plaques of the foliated chart. Two codimension q foliated charts (φ_1, U_1) and (φ_2, U_2) are called compatible if there exists a diffeomorphism $\psi_{12} : \varphi_1(U_1 \cap U_2) \to \varphi_2(U_1 \cap U_2)$ such that the following diagram commutes

$$\begin{tikzcd} & U_1 \cap U_2 \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\ \varphi_1(U_1 \cap U_2) \arrow[rr, "\psi_{12}"] & & \varphi_2(U_1 \cap U_2) \end{tikzcd} \tag{6.1}$$

A foliated atlas on a manifold $\overline{M}$ is a collection $\{(\varphi_\alpha, U_\alpha)\}_\alpha$ of foliated charts that are mutually compatible and such that $\overline{M} = \bigcup_\alpha U_\alpha$.

The compatibility of two foliated charts (φ_1, U_1) and (φ_2, U_2) is defined so that it ensures that, when restricted to $U_1 \cap U_2$, both submersions φ_1 and φ_2 have the same plaques. This implies that the following is an equivalence relation in $\overline{M}$.

$$x \sim y \iff \begin{array}{l} \text{there is a sequence of plaques } (P_k)_{k=0}^l \text{ for foliated charts} \\ (\varphi_k, U_k)_{k=0}^l, \text{respectively, of the foliated atlas, such that } x \in P_0, \\ y \in P_l, \text{and } P_{k-1} \cap P_k \neq \phi \text{ for every } k = 1, \dots, l \end{array}$$

The equivalence classes are submanifolds of $\overline{M}$ of dimension $\dim(\overline{M}) - q$, where q is the codimension of the foliated charts.

Definition 6.1. A foliation $\mathfrak{F}$ on a manifold $\overline{M}$ is a partition of $\overline{M}$ that is given by the family of equivalence classes of the relation of a foliated atlas. The classes are called the leaves of the foliation.

For a manifold $\overline{M}$ carrying a smooth foliation $\mathfrak{F}$ we denote by $T\mathfrak{F}$ the vector subbundle of $T\overline{M}$ that consists of elements tangent to the leaves of $\mathfrak{F}$. Then, we also denote the associated quotient vector bundle by $T^t\mathfrak{F} = T\overline{M}/T\mathfrak{F}$. The latter will be referred to as the transverse vector bundle of the foliation $\mathfrak{F}$. Since $T^t\mathfrak{F}$ is a smooth vector bundle, we can consider the associated linear frame bundle, which we will denote by $L_T(\mathfrak{F})$. In particular, $L_T(\mathfrak{F})$ is a principal fiber bundle with structure group $\mathrm{GL}_q(\mathbb{R})$. The principal bundle $L_T(\mathfrak{F})$ is called the transverse frame bundle since it allows us to study the geometry transverse to the foliation $\mathfrak{F}$.

The study of the transverse geometry of a foliation $\mathfrak{F}$ is based on a natural foliation in $L_T(\mathfrak{F})$, that is defined as follows.

Suppose that for a foliation $\mathfrak{F}$ on a manifold $\overline{M}$ we choose a foliated atlas $\{(\varphi_\alpha, U_\alpha)\}_\alpha$ that determines the foliation as in Definition 6.1. For any foliated chart $(\varphi_\alpha, U_\alpha)$ and every $x \in U_\alpha$ we have a linear map $d(\varphi_\alpha)_x : T_x\overline{M} \to \mathbb{R}^q$ whose kernel is $T_x\mathfrak{F}$. This induces a linear isomorphism $d(\varphi_\alpha)^t_x : T^t_x\mathfrak{F} = T_x\overline{M}/T_x\mathfrak{F} \to \mathbb{R}^q$. The latter allows us to define the smooth map

$$\begin{aligned} \varphi^{(1)}_\alpha : L_T(\mathfrak{F}|_{U_\alpha}) &\to L(V_\alpha) \\ A &\mapsto d(\varphi_\alpha)^t_x \circ A, \end{aligned}$$

where $L_T(\mathfrak{F}|_{U_\alpha})$ is the open subset of $L_T(\mathfrak{F})$ given by inverse image of U_α under the natural projection $L_T(\mathfrak{F}) \to \overline{M}$, A is mapped to x under such projection and V_α is the target of φ_α. Next we observe that, since V_α is open in $\mathbb{R}^q$, the manifold $L(V_\alpha)$ is open in $\mathbb{R}^q \times \mathrm{GL}_q(\mathbb{R})$ and so it is open in $\mathbb{R}^{q+q^2}$ as well. Furthermore, from our choices it is easy to check that the commutative diagram (6.1) and the compatibility of charts in a foliated atlas induce a corresponding commutative diagram given by

$$\begin{tikzcd} & L_T(\mathfrak{F}|_{U_1\cap U_2}) \arrow[dl, "\varphi^{(1)}_{\alpha_1}"'] \arrow[dr, "\varphi^{(1)}_{\alpha_2}"] & \\ L(\varphi_{\alpha_1}(U_1\cap U_2)) \arrow[rr, "\psi^{(1)}_{\alpha_1\alpha_2}"] & & L(\varphi_{\alpha_2}(U_1\cap U_2)) \end{tikzcd}$$

where $\psi^{(1)}_{\alpha_1\alpha_2}$ is defined as above for the diffeomorphism $\psi_{\alpha_1\alpha_2}$ for which we have $\varphi_{\alpha_2} = \psi_{\alpha_1\alpha_2}\circ\varphi_{\alpha_1}$, as in diagram (6.1). This shows that the set $\{(\varphi^{(1)}_\alpha, L_T(\mathfrak{F}|_{U_\alpha}))\}_\alpha$ defines a foliated atlas. The corresponding foliation in $L_T(\mathfrak{F})$ is called the lifted foliation. We now have the following fundamental result (see [7]).

Proposition 6.2. *Let $\mathfrak{F}$ be a foliation on a smooth manifold $\overline{M}$. Then, the natural projection $L_T(\mathfrak{F}) \to \overline{M}$ maps the leaves of the lifted foliation of $L_T(\mathfrak{F})$ locally diffeomorphically onto the leaves of $\mathfrak{F}$.*

In order to define transverse geometric structures for a given foliation $\mathfrak{F}$ we consider reductions of $L_T(\mathfrak{F})$ compatible with the lifted foliation. More precisely, we have the following definition which also introduces the notion of a Riemannian foliation.

Definition 6.3. Let $\overline{M}$ be a manifold carrying a smooth foliation $\mathfrak{F}$ of codimension q, and let G be a Lie subgroup of $\mathrm{GL}_q(\mathbb{R})$. A transverse geometric G-structure is a reduction Q of $L_T(\mathfrak{F})$ to the subgroup G that is saturated with respect to the lifted foliation, i.e. such that $Q\cap L \neq \phi$ implies $L \subset Q$ for every leaf L of the lifted foliation. A transverse geometric $O(q)$-structure is also called a transverse Riemannian structure. A foliation endowed with a transverse Riemannian structure is called a Riemannian foliation.

Hence, a transverse Riemannian structure defines a Riemannian metric on the bundle $T\overline{M}/T\mathfrak{F} = T^t\mathfrak{F}$. However, a transverse Riemannian structure is more than a simple Riemannian metric on $T^t\mathfrak{F}$. Since the $O(q)$-reduction that defines a transverse Riemannian structure is saturated with respect to the lifted foliation, as in Definition 6.3, then the metric is left invariant as we move along the leaves in $\overline{M}$. This is a well known property of Riemannian foliations whose further discussion can be found in [7] and other books on the subject. Here we observe that, since a Riemannian metric on a manifold defines a distance, the invariance of a transverse Riemannian structure as we move along the leaves can be interpreted as the leaves of the foliation in $\overline{M}$ are equidistant while we move along them. Again, this sort of remark is well known in the theory of foliations and shows that a Riemannian foliation has a distinguished geometry. In particular, not every foliation admits a Riemannian structure, a standard example is given by the Reeb foliation of the sphere S^3 (see [7]).

A fundamental way to construct transverse Riemannian structures for a foliation is to consider suitable Riemannian metrics on the manifold that carries the foliation. To describe such construction we need some additional notions.

For a smooth foliation $\mathfrak{F}$ on a manifold $\overline{M}$, a vector field X on $\overline{M}$ is called foliate if for every vector field Y tangent to the leaves of $\mathfrak{F}$ the vector field $[X, Y]$ is tangent to the leaves as well. In other words, the set of foliate vector fields is the normalizer of the fields tangent to the leaves of $\mathfrak{F}$ in the Lie algebra of all vector fields on $\overline{M}$. A Riemannian metric h in $\overline{M}$ is called bundle-like for the foliation $\mathfrak{F}$ if the real-valued function $h(X, Y)$ is constant along the leaves of $\mathfrak{F}$ for every pair of vector fields X, Y that are foliate and perpendicular to $T\mathfrak{F}$ with respect to h.

Suppose that h is a Riemannian metric on a manifold $\overline{M}$ and that $\mathfrak{F}$ is a foliation on $\overline{M}$. Then, the canonical projection $T\overline{M} \to T^t\mathfrak{F}$ allows us to induce a Riemannian metric on the bundle $T^t\mathfrak{F}$, that in turn provides an $O(q)$-reduction of the transverse frame bundle $L_T(\mathfrak{F})$ (where q is the codimension of $\mathfrak{F}$). Nevertheless, such reduction does not necessarily define a transverse Riemannian structure. The next result states that bundle-like metrics are precisely those that define transverse Riemannian structures. The proof of this theorem can be found in [7].

Proposition 6.4. *Let $\overline{M}$ be a manifold carrying a smooth foliation $\mathfrak{F}$ of codimension q. For every Riemannian metric h on $\overline{M}$, denote by $O_T(\overline{M}, h)$ the $O(q)$-reduction of $L_T(\mathfrak{F})$ given by the Riemannian metric on $T^t(\mathfrak{F})$ coming from h and the natural projection $T\overline{M} \to T^t\mathfrak{F}$. If h is a bundle-like metric, then $O_T(\overline{M}, h)$ defines a transverse Riemannian structure on $\mathfrak{F}$. Conversely, for every transverse Riemannian structure given by a reduction Q as in Definition* 6.3*, there is a bundle-like metric h on $\overline{M}$ such that $Q = O_T(\overline{M}, h)$.*

We say that a bundle-like metric h on $\overline{M}$ is compatible with the Riemannian foliation if $O_T(\overline{M}, h)$ is the reduction that defines the corresponding transverse Riemannian structure.

A fundamental property of Riemannian foliations is that, with respect to compatible bundle-like metrics, geodesics that start perpendicular to a leaf of the foliation stay perpendicular to all leaves.

Proposition 6.5. *Let $\mathfrak{F}$ be a Riemannian foliation on a manifold $\overline{M}$ and let h be a compatible bundle-like metric. If γ is a geodesic of h such that $\gamma'(t_0) \in (T_{\gamma(t_0)}\mathfrak{F})^\perp$, for some t_0, then $\gamma'(t) \in (T_{\gamma(t)}\mathfrak{F})^\perp$ for every t.*

This result is fundamental in the theory of Riemannian foliations and its proof can be found in [7]. We can provide its geometric interpretation as follows. Let $\overline{M}$, $\mathfrak{F}$ and h be as in Proposition 6.5, and denote by $T\mathfrak{F}^\perp$ the orthogonal complement of $T\mathfrak{F}$ in $T\overline{M}$; in particular, $T\overline{M} = T\mathfrak{F} \oplus T\mathfrak{F}^\perp$. Hence, Proposition 6.5 states that every geodesic with an initial velocity vector in $T\mathfrak{F}^\perp$ has velocity vectors contained in $T\mathfrak{F}^\perp$ for all time.

In a sense, the above states that the orthogonal complement $T\mathfrak{F}^\perp$ contains all geodesics perpendicular to $T\mathfrak{F}$. If the codimension of $\mathfrak{F}$ is 1, then $T\mathfrak{F}^\perp$ is one-dimensional and it can be integrated to a smooth one-dimensional foliation $\mathfrak{F}^\perp$ whose leaves are perpendicular to those of $\mathfrak{F}$. In such case, Proposition 6.5 ensures that the leaves of $\mathfrak{F}^\perp$ are geodesics with respect to the bundle-like metric h.

If $\mathfrak{F}$ has codimension greater than 1, then we can still consider the possibility of $T\mathfrak{F}^\perp$ being integrable, e.g. to satisfy the hypothesis of Frobenius theorem. If $T\mathfrak{F}^\perp$ is indeed integrable, we do have a foliation $\mathfrak{F}^\perp$ whose leaves are orthogonal to those of $\mathfrak{F}$. Again, in this case, Proposition 6.5 implies that the leaves of $\mathfrak{F}^\perp$ are totally geodesic. More precisely, we have the following well known result in the theory of foliations.

Proposition 6.6. *Let $\mathfrak{F}$ be a Riemannian foliation on a manifold $\overline{M}$ and let h be a compatible bundle-like metric. If the vector bundle $T\mathfrak{F}^\perp$ is integrable, then its integral submanifolds define a totally geodesic foliation, i.e. a foliation for which every leaf is a totally geodesic submanifold of $\overline{M}$ with respect to the metric h.*

Our interest in the geometry of foliations comes from our study of actions of Lie groups, where an action of the group partitions the manifold on orbits. We will see that such partitions define foliations with distinguished geometry for suitable actions. In what follows $\overline{M}$ will denote a smooth manifold and G a connected Lie group acting smoothly on the left on $\overline{M}$. For such a G-action, we will denote the stabilizer of a point $x \in \overline{M}$ by G_x. Then, the action of G on $\overline{M}$ is called free (locally free) if for every $x \in \overline{M}$ the stabilizer G_x is trivial (respectively discrete).

A straightforward application of Frobenius theorem on the integrability of vector subbundles of a tangent bundle allows us to obtain the following result (see [15] for a proof). We recall that a parallelism for a vector bundle is a collection of sections whose restriction to every point yields a basis for the fiber at such point.

Proposition 6.7. *If G acts locally freely on $\overline{M}$, then the G-orbits define a smooth foliation on $\overline{M}$. Furthermore, if $X_1, \dots, X_k$ is a basis for the Lie algebra of G, then $X_1^*, \dots, X_k^*$ define a parallelism for the tangent bundle to the G-orbits.*

We now consider the case where G acts locally freely preserving a Riemannian metric on M. A proof of the following well known result can be found in [15].

Proposition 6.8. *If G acts locally freely on $\overline{M}$ preserving a Riemannian metric h, then the G-orbits define a smooth Riemannian foliation for which h is a compatible bundle-like metric.*

From this and Propositions 6.5 and 6.6 we obtain the following consequence.

Proposition 6.9. *If G acts locally freely on $\overline{M}$ preserving a Riemannian metric h and γ is a geodesic (with respect to h) perpendicular at some point to a G-orbit, then γ intersects every G-orbit perpendicularly. In particular, if the normal bundle to the G-orbits is integrable, then the integral submanifolds of such normal bundle define a totally geodesic foliation everywhere perpendicular to the G-orbits.*

We now prove the main result of this section: the MASG's of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ provide a pair of foliations with distinguished geometry in the sense discussed above.

Theorem 6.10. *Let H be a MASG in $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ and denote by $\mathrm{H}^n\mathbb{C}_H$ the set of points with trivial stabilizer. Then, $\mathrm{H}^n\mathbb{C}_H$ is a connected open conull H-invariant subset of $\mathrm{H}^n\mathbb{C}$ on which the collection of H-orbits defines a smooth foliation $\mathcal{O}$ that satisfies the following properties*

1. *$\mathcal{O}$ is a Lagrangian and Riemannian foliation with complete flat parallel leaves.*
2. *The orthogonal complement $T\mathcal{O}^\perp$ of the tangent bundle to $\mathcal{O}$ is integrable and thus defines a foliation $\mathfrak{F}$.*
3. *The foliation $\mathfrak{F}$ is Lagrangian and totally geodesic.*

Also, at every point of $\mathrm{H}^n\mathbb{C}_H$ there is a local coordinate system $(x_1, \dots, x_n, y_1, \dots, y_n)$ such that the restriction of $(x_1, \dots, x_n)$ (of $(y_1, \dots, y_n)$, respectively) to a leaf of $\mathcal{O}$ (to a leaf of $\mathfrak{F}$, respectively) defines a coordinate system on such a leaf.

Proof. The claim about the properties of the subset $\mathrm{H}^n\mathbb{C}_H$ follows from the last claim in Theorem 5.7.

By Theorem 5.5 and Proposition 6.8 we have that $\mathcal{O}$ is both Lagrangian and Riemannian. The statement that the leaves of $\mathcal{O}$ are flat and parallel follows from Proposition 5.1 and Theorem 5.6, respectively. The completeness of the leaves follows from their homogeneity.

To prove the integrability of $T\mathcal{O}^\perp$, let $X_1, \dots, X_n$ be a basis for the Lie algebra of H. By Theorem 6.7, the vector fields $X_1^*, \dots, X_n^*$ define a parallelism of the vector bundle $T\mathcal{O}$. Since $T\mathcal{O}^\perp = iT\mathcal{O}$, it follows that $iX_1^*, \dots, iX_n^*$ is a parallelism for $T\mathcal{O}^\perp$. The vector fields X_j commute with each other because H is Abelian, and from this one can easily see that the vector fields iX_j^* commute with each other as well. We thus have concluded the existence of a parallelism for $T\mathcal{O}^\perp$ consisting of commuting vector fields. Hence the integrability of $T\mathcal{O}^\perp$ follows from Frobenius theorem.

Once we know that $T\mathcal{O}^\perp$ is integrable to a foliation $\mathfrak{F}$, it follows that such $\mathfrak{F}$ is a Lagrangian foliation by the corresponding property of $\mathcal{O}$. Also, $\mathfrak{F}$ is totally geodesic as a consequence of Proposition 6.9.

Finally, the existence of the required coordinate system is obtained from the vector fields X_j^* and iX_k^* as follows. By the proof of Frobenius theorem as found in [17] it follows that if $Z_1, \dots, Z_l$ is a parallelism of a vector subbundle (of the tangent bundle of a manifold) such that $[Z_j, Z_k] = 0$ for all j, k, then not just the vector subbundle is integrable, but one can find functions $(x_1, \dots, x_l)$ (locally defined on the ambient manifold) whose restriction to each integral submanifold defines a coordinate system in it that satisfies $Z_j = \frac{\partial}{\partial x_j}$. The last claim is thus obtained from this remark by using that the vector fields X_j^* and iX_k^*, for $j, k = 1, \dots, n$ commute with each other. $\square$

The previous result shows that from an action of a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ we obtain a pair of mutually orthogonal foliations $(\mathcal{O}, \mathfrak{F})$ that have distinguished geometric properties. We can think of each such pair as providing a sort of Lagrangian frame on the complex hyperbolic space $\mathrm{H}^n\mathbb{C}$. Such frame yields the geometric foundation for the symbols that define the commutative C^*-algebras of Toeplitz operators introduced in the first part of this work [16]. This frame even comes with local coordinates that are thus adapted to the corresponding symbols. Furthermore, from the definition of such symbols, the leaves of the foliation $\mathcal{O}$ play the role of their level sets and the leaves of the foliation $\mathfrak{F}$ thus play the role of the corresponding gradient sets, so to speak.

Given the relevance of the properties of the pair of foliations obtained in Theorem 6.10 we consider the following definition.

Definition 6.11. A Lagrangian frame in $\mathrm{H}^n\mathbb{C}$ is a pair of foliations $(\mathfrak{F}_1, \mathfrak{F}_2)$ defined in a connected open conull subset U of $\mathrm{H}^n\mathbb{C}$ that satisfies the following conditions

1. Both $\mathfrak{F}_1$ and $\mathfrak{F}_2$ are Lagrangian foliations and perpendicular to each other wherever they meet.
2. The foliation $\mathfrak{F}_1$ is Riemannian with complete flat parallel leaves.
3. The foliation $\mathfrak{F}_2$ is totally geodesic.

In this case, we say that the Lagrangian frame is defined in U.

From our results up to this point it is possible to show that every Lagrangian frame is in fact always essentially given by a MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$. More precisely, we have the next result. We recall that the restriction of a foliation $\mathfrak{F}$ defined on a manifold $\overline{M}$ to an open subset U is the foliation whose leaves are the connected components of the intersection with U of the leaves of $\mathfrak{F}$.

Theorem 6.12. *Let $(\mathfrak{F}_1, \mathfrak{F}_2)$ be a Lagrangian frame defined in U, a connected open conull subset of $\mathrm{H}^n\mathbb{C}$. Then there is a MASG H of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ such that if we denote by $(\mathcal{O}_H, \mathfrak{F}_H)$ the Lagrangian frame defined by H from Theorem 6.10, the following are satisfied*

1. *The subset U is contained in $\mathrm{H}^n\mathbb{C}_H$ (as defined in Theorem 6.10).*
2. *The foliation $\mathfrak{F}_1$ is the restriction of $\mathcal{O}_H$ to U.*
3. *The foliation $\mathfrak{F}_2$ is the restriction of $\mathfrak{F}_H$ to U.*

Proof. Let L be a leaf of $\mathfrak{F}_1$. Then, Theorem 5.9 implies that there is a MASG H of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ such that L is an H-orbit. As above, let us denote by $(\mathcal{O}_H, \mathfrak{F}_H)$ the Lagrangian frame defined by H. By Theorem 6.10 and the definition of a Lagrangian frame, both $\mathfrak{F}_H$ and $\mathfrak{F}_2$ are totally geodesic and perpendicular to L at the intersection points with it. Hence, the leaves of both $\mathfrak{F}_H$ and $\mathfrak{F}_2$ are given in a neighborhood of L as the images under the exponential map of the fibers of the normal bundle to L. We conclude that the leaves of $\mathfrak{F}_H$ and $\mathfrak{F}_2$ coincide in a neighborhood of L, and this neighborhood is contained in $U \cap \mathrm{H}^n\mathbb{C}_H$. To complete the proof, we consider the set of points $x \in U$ that belong to a leaf of $\mathfrak{F}_1$ which is an H-orbit. We note that the H-orbits in $U \cap \mathrm{H}^n\mathbb{C}_H$ are characterized as (real) n-dimensional submanifolds perpendicular to $\mathfrak{F}_2$. Using this fact, a standard argument implies that, by the connectedness of U, the foliation $\mathfrak{F}_1$ consists of H-orbits and so it is the restriction of $\mathcal{O}_H$. Moreover, the H-action is necessarily free on U and so $U \subset \mathrm{H}^n\mathbb{C}_H$. Finally, $\mathfrak{F}_2$ is the restriction of $\mathfrak{F}_H$ since the leaves of both are integral submanifolds of the normal bundle to the H-orbits. □

We note that in some deep sense a Lagrangian frame, as a pair of foliations with certain specific properties, is an appropriate multidimensional analog of a pencil of hyperbolic geodesics, consisting of cycles and geodesics, on the unit disk as considered in [3].

7. Commutative algebras of Toeplitz operators and Lagrangian frames.

In the preceding sections we have shown that the MASG's of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ define, through their orbits, geometric objects with distinguished properties. As the following result shows such study of the geometry of the MASG's of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ is fundamental to understand the structure of the commutative C^*-algebras generated by Toeplitz operators.

Theorem 7.1. *For a subspace $\mathcal{A}$ of $L_\infty(\mathbb{B}^n) \cap C^\infty(\mathbb{B}^n)$ the following conditions are equivalent.*

1. *There is a Lagrangian frame $(\mathfrak{F}_1, \mathfrak{F}_2)$ defined in a connected open conull subset U of $\mathbb{B}^n$ such that if $a \in \mathcal{A}$, then every level set of a is saturated with respect to the foliation $\mathfrak{F}_1$, i.e., every such level set is a union of leaves of $\mathfrak{F}_1$.*
2. *There is an Abelian subgroup H of either $\mathrm{Aut}(\mathbb{B}^n)$ or $\mathrm{Aut}(D_n)$ listed in Section 2 and a biholomorphism φ from $\mathbb{B}^n$ onto either $\mathbb{B}^n$ or D_n, correspondingly, such that $\mathcal{A} \subset \varphi^*(\mathcal{A}_H) = \{a \circ \varphi : a \in \mathcal{A}_H\}$, where $\mathcal{A}_H$ is the subspace of either $L_\infty(\mathbb{B}^n)$ or $L_\infty(D_n)$, correspondingly, consisting of all H-invariant functions.*

Proof. That (2) implies (1) is the content of Theorem 6.10 together with the classification of MASG of $\mathrm{Aut}(\mathrm{H}^n\mathbb{C})$ given in Theorem 3.6.

To prove that (1) implies (2) we use again the classification from Theorem 3.6 as well as Theorem 6.12. From these two results it follows that there is a subgroup H of either $\mathrm{Aut}(\mathbb{B}^n)$ or $\mathrm{Aut}(D_n)$ listed in Section 2 such that, up to a biholomorphism, the Lagrangian frame $(\mathfrak{F}_1, \mathfrak{F}_2)$ is a restriction of $(\mathcal{O}_H, \mathfrak{F}_H)$ as defined in Theorem 6.12. Hence, on the subset U every level subset of every $a \in \mathcal{A}$ is saturated with respect to the foliation $\mathcal{O}_H$. This implies that every $a \in \mathcal{A}$ is H-invariant in U. Hence, the result follows by the density of U that comes from the fact that it is conull. □

Corollary 7.2. *Given any Lagrangian frame $\mathfrak{F} = (\mathfrak{F}_1, \mathfrak{F}_2)$ on the unit ball $\mathbb{B}^n$, denote by $\mathcal{A}_{\mathfrak{F}}$ the set of all $L_\infty(\mathbb{B}^n)$-functions constant on leaves of $\mathfrak{F}_1$. Then the C^*-algebra generated by Toeplitz operators with symbols from $\mathcal{A}_{\mathfrak{F}}$ is commutative on each weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$, $\lambda \in (-1, \infty)$, considered in* [16].

Furthermore, each such commutative Toeplitz operator algebra is unitary equivalent to one considered in [16].

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